SIGNALS & SYSTEMS

SECOND EDITION
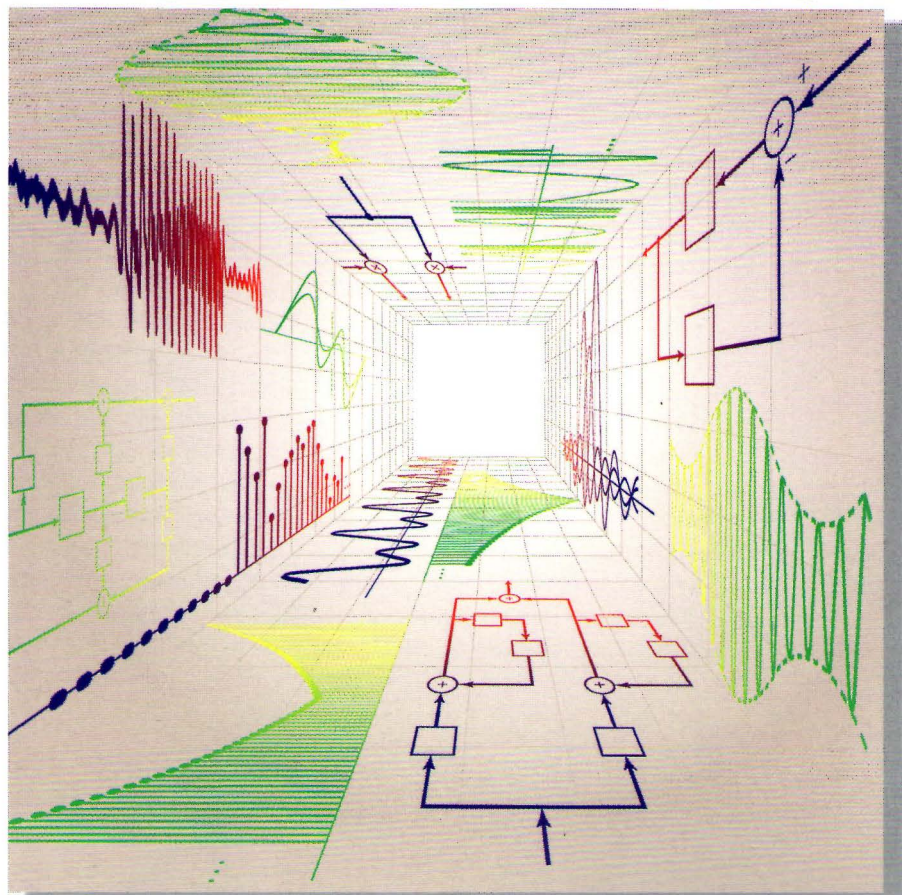
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WITH S. HAMID NAWAB

SECOND EDITION

SIGNALS
&
SYSTEMS

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PREFACE

This book is the second edition of a text designed for undergraduate courses in signals and systems. While such courses are frequently found in electrical engineering curricula, the concepts and techniques that form the core of the subject are of fundamental importance in all engineering disciplines. In fact, the scope of potential and actual applications of the methods of signal and system analysis continues to expand as engineers are confronted with new challenges involving the synthesis or analysis of complex processes. For these reasons we feel that a course in signals and systems not only is an essential element in an engineering program but also can be one of the most rewarding, exciting, and useful courses that engineering students take during their undergraduate education.

Our treatment of the subject of signals and systems in this second edition maintains the same general philosophy as in the first edition but with significant rewriting, restructuring, and additions. These changes are designed to help both the instructor in presenting the subject material and the student in mastering it. In the preface to the first edition we stated that our overall approach to signals and systems had been guided by the continuing developments in technologies for signal and system design and implementation, which made it increasingly important for a student to have equal familiarity with techniques suitable for analyzing and synthesizing both continuous-time and discrete-time systems. As we write the preface to this second edition, that observation and guiding principle are even more true than before. Thus, while students studying signals and systems should certainly have a solid foundation in disciplines based on the laws of physics, they must also have a firm grounding in the use of computers for the analysis of phenomena and the implementation of systems and algorithms. As a consequence, engineering curricula now reflect a blend of subjects, some involving continuous-time models and others focusing on the use of computers and discrete representations. For these reasons, signals and systems courses that bring discrete-time and continuous-time concepts together in a unified way play an increasingly important role in the education of engineering students and in their preparation for current and future developments in their chosen fields.

It is with these goals in mind that we have structured this book to develop in parallel the methods of analysis for continuous-time and discrete-time signals and systems. This approach also offers a distinct and extremely important pedagogical advantage. Specifically, we are able to draw on the similarities between continuous- and discrete-time methods in order to share insights and intuition developed in each domain. Similarly, we can exploit the differences between them to sharpen an understanding of the distinct properties of each.

In organizing the material both originally and now in the second edition, we have also considered it essential to introduce the student to some of the important uses of the basic methods that are developed in the book. Not only does this provide the student with an appreciation for the range of applications of the techniques being learned and for directions for further study, but it also helps to deepen understanding of the subject. To achieve this goal we include introductory treatments on the subjects of filtering, commu-

nications, sampling, discrete-time processing of continuous-time signals, and feedback. In fact, in one of the major changes in this second edition, we have introduced the concept of frequency-domain filtering very early in our treatment of Fourier analysis in order to provide both motivation for and insight into this very important topic. In addition, we have again included an up-to-date bibliography at the end of the book in order to assist the student who is interested in pursuing additional and more advanced studies of the methods and applications of signal and system analysis.

The organization of the book reflects our conviction that full mastery of a subject of this nature cannot be accomplished without a significant amount of practice in using and applying the tools that are developed. Consequently, in the second edition we have significantly increased the number of worked examples within each chapter. We have also enhanced one of the key assets of the first edition, namely the end-of-chapter homework problems. As in the first edition, we have included a substantial number of problems, totaling more than 600 in number. A majority of the problems included here are new and thus provide additional flexibility for the instructor in preparing homework assignments. In addition, in order to enhance the utility of the problems for both the student and the instructor we have made a number of other changes to the organization and presentation of the problems. In particular, we have organized the problems in each chapter under several specific headings, each of which spans the material in the entire chapter but with a different objective. The first two sections of problems in each chapter emphasize the mechanics of using the basic concepts and methods presented in the chapter. For the first of these two sections, which has the heading Basic Problems with Answers, we have also provided answers (but not solutions) at the end of the book. These answers provide a simple and immediate way for the student to check his or her understanding of the material. The problems in this first section are generally appropriate for inclusion in homework sets. Also, in order to give the instructor additional flexibility in assigning homework problems, we have provided a second section of Basic Problems for which answers have not been included.

A third section of problems in each chapter, organized under the heading of Advanced Problems, is oriented toward exploring and elaborating upon the foundations and practical implications of the material in the text. These problems often involve mathematical derivations and more sophisticated use of the concepts and methods presented in the chapter. Some chapters also include a section of Extension Problems which involve extensions of material presented in the chapter and/or involve the use of knowledge from applications that are outside the scope of the main text (such as advanced circuits or mechanical systems). The overall variety and quantity of problems in each chapter will hopefully provide students with the means to develop their understanding of the material and instructors with considerable flexibility in putting together homework sets that are tailored to the specific needs of their students. A solutions manual is also available to instructors through the publisher.

Another significant additional enhancement to this second edition is the availability of the companion book *Explorations in Signals and Systems Using MATLAB* by Buck, Daniel, and Singer. This book contains MATLAB®-based computer exercises for each topic in the text, and should be of great assistance to both instructor and student.

Students using this book are assumed to have a basic background in calculus as well as some experience in manipulating complex numbers and some exposure to differential

equations. With this background, the book is self-contained. In particular, no prior experience with system analysis, convolution, Fourier analysis, or Laplace and z-transforms is assumed. Prior to learning the subject of signals and systems most students will have had a course such as basic circuit theory for electrical engineers or fundamentals of dynamics for mechanical engineers. Such subjects touch on some of the basic ideas that are developed more fully in this text. This background can clearly be of great value to students in providing additional perspective as they proceed through the book.

The Foreword, which follows this preface, is written to offer the reader motivation and perspective for the subject of signals and systems in general and our treatment of it in particular. We begin Chapter 1 by introducing some of the elementary ideas related to the mathematical representation of signals and systems. In particular we discuss transformations (such as time shifts and scaling) of the independent variable of a signal. We also introduce some of the most important and basic continuous-time and discrete-time signals, namely real and complex exponentials and the continuous-time and discrete-time unit step and unit impulse. Chapter 1 also introduces block diagram representations of interconnections of systems and discusses several basic system properties such as causality, linearity and time-invariance. In Chapter 2 we build on these last two properties, together with the sifting property of unit impulses to develop the convolution-sum representation for discrete-time linear, time-invariant (LTI) systems and the convolution integral representation for continuous-time LTI systems. In this treatment we use the intuition gained from our development of the discrete-time case as an aid in deriving and understanding its continuous-time counterpart. We then turn to a discussion of causal, LTI systems characterized by linear constant-coefficient differential and difference equations. In this introductory discussion we review the basic ideas involved in solving linear differential equations (to which most students will have had some previous exposure) and we also provide a discussion of analogous methods for linear difference equations. However, the primary focus of our development in Chapter 2 is not on methods of solution, since more convenient approaches are developed later using transform methods. Instead, in this first look, our intent is to provide the student with some appreciation for these extremely important classes of systems, which will be encountered often in subsequent chapters. Finally, Chapter 2 concludes with a brief discussion of singularity functions—steps, impulses, doublets, and so forth—in the context of their role in the description and analysis of continuous-time LTI systems. In particular, we stress the interpretation of these signals in terms of how they are defined under convolution—that is, in terms of the responses of LTI systems to these idealized signals.

Chapters 3 through 6 present a thorough and self-contained development of the methods of Fourier analysis in both continuous and discrete time and together represent the most significant reorganization and revision in the second edition. In particular, as we indicated previously, we have introduced the concept of frequency-domain filtering at a much earlier point in the development in order to provide motivation for and a concrete application of the Fourier methods being developed. As in the first edition, we begin the discussions in Chapter 3 by emphasizing and illustrating the two fundamental reasons for the important role Fourier analysis plays in the study of signals and systems in both continuous and discrete time: (1) extremely broad classes of signals can be represented as weighted sums or integrals of complex exponentials; and (2) the response of an LTI system to a complex exponential input is the same exponential multiplied by a complex-

number characteristic of the system. However, in contrast to the first edition, the focus of attention in Chapter 3 is on Fourier series representations for periodic signals in both continuous time and discrete time. In this way we not only introduce and examine many of the properties of Fourier representations without the additional mathematical generalization required to obtain the Fourier transform for aperiodic signals, but we also can introduce the application to filtering at a very early stage in the development. In particular, taking advantage of the fact that complex exponentials are eigenfunctions of LTI systems, we introduce the frequency response of an LTI system and use it to discuss the concept of frequency-selective filtering, to introduce ideal filters, and to give several examples of nonideal filters described by differential and difference equations. In this way, with a minimum of mathematical preliminaries, we provide the student with a deeper appreciation for what a Fourier representation means and why it is such a useful construct.

Chapters 4 and 5 then build on the foundation provided by Chapter 3 as we develop first the continuous-time Fourier transform in Chapter 4 and, in a parallel fashion, the discrete-time Fourier transform in Chapter 5. In both chapters we derive the Fourier transform representation of an aperiodic signal as the limit of the Fourier series for a signal whose period becomes arbitrarily large. This perspective emphasizes the close relationship between Fourier series and transforms, which we develop further in subsequent sections and which allows us to transfer the intuition developed for Fourier series in Chapter 3 to the more general context of Fourier transforms. In both chapters we have included a discussion of the many important properties of Fourier transforms, with special emphasis placed on the convolution and multiplication properties. In particular, the convolution property allows us to take a second look at the topic of frequency-selective filtering, while the multiplication property serves as the starting point for our treatment of sampling and modulation in later chapters. Finally, in the last sections in Chapters 4 and 5 we use transform methods to determine the frequency responses of LTI systems described by differential and difference equations and to provide several examples illustrating how Fourier transforms can be used to compute the responses for such systems. To supplement these discussions (and later treatments of Laplace and z-transforms) we have again included an Appendix at the end of the book that contains a description of the method of partial fraction expansion.

Our treatment of Fourier analysis in these two chapters is characteristic of the parallel treatment we have developed. Specifically, in our discussion in Chapter 5, we are able to build on much of the insight developed in Chapter 4 for the continuous-time case, and toward the end of Chapter 5 we emphasize the complete duality in continuous-time and discrete-time Fourier representations. In addition, we bring the special nature of each domain into sharper focus by contrasting the differences between continuous- and discrete-time Fourier analysis.

As those familiar with the first edition will note, the lengths and scopes of Chapters 4 and 5 in the second edition are considerably smaller than their first edition counterparts. This is due not only to the fact that Fourier series are now dealt with in a separate chapter but also to our moving several topics into Chapter 6. The result, we believe, has several significant benefits. First, the presentation in three shorter chapters of the basic concepts and results of Fourier analysis, together with the introduction of the concept of frequency-selective filtering, should help the student in organizing his or her understanding of this material and in developing some intuition about the frequency domain and appreciation for its potential applications. Then, with Chapters 3-5 as a foundation, we can engage in

a more detailed look at a number of important topics and applications. In Chapter 6 we take a deeper look at both the time- and frequency-domain characteristics of LTI systems. For example, we introduce magnitude-phase and Bode plot representations for frequency responses and discuss the effect of frequency response phase on the time domain characteristics of the output of an LTI system. In addition, we examine the time- and frequency-domain behavior of ideal and nonideal filters and the tradeoffs between these that must be addressed in practice. We also take a careful look at first- and second-order systems and their roles as basic building blocks for more complex system synthesis and analysis in both continuous and discrete time. Finally, we discuss several other more complex examples of filters in both continuous and discrete time. These examples together with the numerous other aspects of filtering explored in the problems at the end of the chapter provide the student with some appreciation for the richness and flavor of this important subject. While each of the topics in Chapter 6 was present in the first edition, we believe that by reorganizing and collecting them in a separate chapter following the basic development of Fourier analysis, we have both simplified the introduction of this important topic in Chapters 3-5 and presented in Chapter 6 a considerably more cohesive picture of time- and frequency-domain issues.

In response to suggestions and preferences expressed by many users of the first edition we have modified notation in the discussion of Fourier transforms to be more consistent with notation most typically used for continuous-time and discrete-time Fourier transforms. Specifically, beginning with Chapter 3 we now denote the continuous-time Fourier transform as $X(j\omega)$ and the discrete-time Fourier transform as $X(e^{j\omega})$. As with all options with notation, there is not a unique best choice for the notation for Fourier transforms. However, it is our feeling, and that of many of our colleagues, that the notation used in this edition represents the preferable choice.

Our treatment of sampling in Chapter 7 is concerned primarily with the sampling theorem and its implications. However, to place this subject in perspective we begin by discussing the general concepts of representing a continuous-time signal in terms of its samples and the reconstruction of signals using interpolation. After using frequency-domain methods to derive the sampling theorem, we consider both the frequency and time domains to provide intuition concerning the phenomenon of aliasing resulting from undersampling. One of the very important uses of sampling is in the discrete-time processing of continuous-time signals, a topic that we explore at some length in this chapter. Following this, we turn to the sampling of discrete-time signals. The basic result underlying discrete-time sampling is developed in a manner that parallels that used in continuous time, and the applications of this result to problems of decimation and interpolation are described. Again a variety of other applications, in both continuous and discrete time, are addressed in the problems.

Once again the reader acquainted with our first edition will note a change, in this case involving the reversal in the order of the presentation of sampling and communications. We have chosen to place sampling before communications in the second edition both because we can call on simple intuition to motivate and describe the processes of sampling and reconstruction from samples and also because this order of presentation then allows us in Chapter 8 to talk more easily about forms of communication systems that are closely related to sampling or rely fundamentally on using a sampled version of the signal to be transmitted.

Our treatment of communications in Chapter 8 includes an in-depth discussion of continuous-time sinusoidal amplitude modulation (AM), which begins with the straight-forward application of the multiplication property to describe the effect of sinusoidal AM in the frequency domain and to suggest how the original modulating signal can be recovered. Following this, we develop a number of additional issues and applications related to sinusoidal modulation, including frequency-division multiplexing and single-sideband modulation. Many other examples and applications are described in the problems. Several additional topics are covered in Chapter 8. The first of these is amplitude modulation of a pulse train and time-division multiplexing, which has a close connection to the topic of sampling in Chapter 7. Indeed we make this tie even more explicit and provide a look into the important field of digital communications by introducing and briefly describing the topics of pulse-amplitude modulation (PAM) and intersymbol interference. Finally, our discussion of frequency modulation (FM) provides the reader with a look at a nonlinear modulation problem. Although the analysis of FM systems is not as straightforward as for the AM case, our introductory treatment indicates how frequency-domain methods can be used to gain a significant amount of insight into the characteristics of FM signals and systems. Through these discussions and the many other aspects of modulation and communications explored in the problems in this chapter we believe that the student can gain an appreciation both for the richness of the field of communications and for the central role that the tools of signals and systems analysis play in it.

Chapters 9 and 10 treat the Laplace and z-transforms, respectively. For the most part, we focus on the bilateral versions of these transforms, although in the last section of each chapter we discuss unilateral transforms and their use in solving differential and difference equations with nonzero initial conditions. Both chapters include discussions on: the close relationship between these transforms and Fourier transforms; the class of rational transforms and their representation in terms of poles and zeros; the region of convergence of a Laplace or z-transform and its relationship to properties of the signal with which it is associated; inverse transforms using partial fraction expansion; the geometric evaluation of system functions and frequency responses from pole-zero plots; and basic transform properties. In addition, in each chapter we examine the properties and uses of system functions for LTI systems. Included in these discussions are the determination of system functions for systems characterized by differential and difference equations; the use of system function algebra for interconnections of LTI systems; and the construction of cascade, parallel- and direct-form block-diagram representations for systems with rational system functions.

The tools of Laplace and z-transforms form the basis for our examination of linear feedback systems in Chapter 11. We begin this chapter by describing a number of the important uses and properties of feedback systems, including stabilizing unstable systems, designing tracking systems, and reducing system sensitivity. In subsequent sections we use the tools that we have developed in previous chapters to examine three topics that are of importance for both continuous-time and discrete-time feedback systems. These are root locus analysis, Nyquist plots and the Nyquist criterion, and log-magnitude/phase plots and the concepts of phase and gain margins for stable feedback systems.

The subject of signals and systems is an extraordinarily rich one, and a variety of approaches can be taken in designing an introductory course. It was our intention with the first edition and again with this second edition to provide instructors with a great deal of

flexibility in structuring their presentations of the subject. To obtain this flexibility and to maximize the usefulness of this book for instructors, we have chosen to present thorough, in-depth treatments of a cohesive set of topics that forms the core of most introductory courses on signals and systems. In achieving this depth we have of necessity omitted introductions to topics such as descriptions of random signals and state space models that are sometimes included in first courses on signals and systems. Traditionally, at many schools, such topics are not included in introductory courses but rather are developed in more depth in follow-on undergraduate courses or in courses explicitly devoted to their investigation. Although we have not included an introduction to state space in the book, instructors of introductory courses can easily incorporate it into the treatments of differential and difference equations that can be found throughout the book. In particular, the discussions in Chapters 9 and 10 on block diagram representations for systems with rational system functions and on unilateral transforms and their use in solving differential and difference equations with initial conditions form natural points of departure for the discussions of state-space representations.

A typical one-semester course at the sophomore-junior level using this book would cover Chapters 1-5 in reasonable depth (although various topics in each chapter are easily omitted at the discretion of the instructor) with selected topics chosen from the remaining chapters. For example, one possibility is to present several of the basic topics in Chapters 6-8 together with a treatment of Laplace and z-transforms and perhaps a brief introduction to the use of system function concepts to analyze feedback systems. A variety of alternate formats are possible, including one that incorporates an introduction to state space or one in which more focus is placed on continuous-time systems by de-emphasizing Chapters 5 and 10 and the discrete-time topics in Chapters 3, 7, 8, and 11.

In addition to these course formats this book can be used as the basic text for a thorough, two-semester sequence on linear systems. Alternatively, the portions of the book not used in a first course on signals and systems can, together with other sources, form the basis for a subsequent course. For example, much of the material in this book forms a direct bridge to subjects such as state space analysis, control systems, digital signal processing, communications and statistical signal processing. Consequently, a follow-on course can be constructed that uses some of the topics in this book together with supplementary material in order to provide an introduction to one or more of these advanced subjects. In fact, a new course following this model has been developed at MIT and has proven not only to be a popular course among our students but also a crucial component of our signals and systems curriculum.

As it was with the first edition, in the process of writing this book we have been fortunate to have received assistance, suggestions, and support from numerous colleagues, students and friends. The ideas and perspectives that form the heart of this book have continued to evolve as a result of our own experiences in teaching signals and systems and the influences of the many colleagues and students with whom we have worked. We would like to thank Professor Ian T. Young for his contributions to the first edition of this book and to thank and welcome Professor Hamid Nawab for the significant role he played in the development and complete restructuring of the examples and problems for this second edition. We also express our appreciation to John Buck, Michael Daniel and Andrew Singer for writing the MATLAB companion to the text. In addition, we would like to thank Jason Oppenheim for the use of one of his original photographs and Vivian Berman

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The encouragement, patience, technical support, and enthusiasm provided by Prentice-Hall, and in particular by Marcia Horton, Tom Robbins, Don Fowley, and their predecessors and by Ralph Pescatore of TKM Productions and the production staff at Prentice-Hall, have been crucial in making this second edition a reality.

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FOREWORD

The concepts of signals and systems arise in a wide variety of fields, and the ideas and techniques associated with these concepts play an important role in such diverse areas of science and technology as communications, aeronautics and astronautics, circuit design, acoustics, seismology, biomedical engineering, energy generation and distribution systems, chemical process control, and speech processing. Although the physical nature of the signals and systems that arise in these various disciplines may be drastically different, they all have two very basic features in common. The signals, which are functions of one or more independent variables, contain information about the behavior or nature of some phenomenon, whereas the systems respond to particular signals by producing other signals or some desired behavior. Voltages and currents as a function of time in an electrical circuit are examples of signals, and a circuit is itself an example of a system, which in this case responds to applied voltages and currents. As another example, when an automobile driver depresses the accelerator pedal, the automobile responds by increasing the speed of the vehicle. In this case, the system is the automobile, the pressure on the accelerator pedal the input to the system, and the automobile speed the response. A computer program for the automated diagnosis of electrocardiograms can be viewed as a system which has as its input a digitized electrocardiogram and which produces estimates of parameters such as heart rate as outputs. A camera is a system that receives light from different sources and reflected from objects and produces a photograph. A robot arm is a system whose movements are the response to control inputs.

In the many contexts in which signals and systems arise, there are a variety of problems and questions that are of importance. In some cases, we are presented with a specific system and are interested in characterizing it in detail to understand how it will respond to various inputs. Examples include the analysis of a circuit in order to quantify its response to different voltage and current sources and the determination of an aircraft's response characteristics both to pilot commands and to wind gusts.

In other problems of signal and system analysis, rather than analyzing existing systems, our interest may be focused on designing systems to process signals in particular ways. One very common context in which such problems arise is in the design of systems to enhance or restore signals that have been degraded in some way. For example, when a pilot is communicating with an air traffic control tower, the communication can be degraded by the high level of background noise in the cockpit. In this and many similar cases, it is possible to design systems that will retain the desired signal, in this case the pilot's voice, and reject (at least approximately) the unwanted signal, i.e., the noise. A similar set of objectives can also be found in the general area of image restoration and image enhancement. For example, images from deep space probes or earth-observing satellites typically represent degraded versions of the scenes being imaged because of limitations of the imaging equipment, atmospheric effects, and errors in signal transmission in returning the images to earth. Consequently, images returned from space are routinely processed by systems to compensate for some of these degradations. In addition, such images are usu-

ally processed to enhance certain features, such as lines (corresponding, for example, to river beds or faults) or regional boundaries in which there are sharp contrasts in color or darkness.

In addition to enhancement and restoration, in many applications there is a need to design systems to extract specific pieces of information from signals. The estimation of heart rate from an electrocardiogram is one example. Another arises in economic forecasting. We may, for example, wish to analyze the history of an economic time series, such as a set of stock market averages, in order to estimate trends and other characteristics such as seasonal variations that may be of use in making predictions about future behavior. In other applications, the focus may be on the design of signals with particular properties. Specifically, in communications applications considerable attention is paid to designing signals to meet the constraints and requirements for successful transmission. For example, long distance communication through the atmosphere requires the use of signals with frequencies in a particular part of the electromagnetic spectrum. The design of communication signals must also take into account the need for reliable reception in the presence of both distortion due to transmission through the atmosphere and interference from other signals being transmitted simultaneously by other users.

Another very important class of applications in which the concepts and techniques of signal and system analysis arise are those in which we wish to modify or control the characteristics of a given system, perhaps through the choice of specific input signals or by combining the system with other systems. Illustrative of this kind of application is the design of control systems to regulate chemical processing plants. Plants of this type are equipped with a variety of sensors that measure physical signals such as temperature, humidity, and chemical composition. The control system in such a plant responds to these sensor signals by adjusting quantities such as flow rates and temperature in order to regulate the ongoing chemical process. The design of aircraft autopilots and computer control systems represents another example. In this case, signals measuring aircraft speed, altitude, and heading are used by the aircraft's control system in order to adjust variables such as throttle setting and the position of the rudder and ailerons. These adjustments are made to ensure that the aircraft follows a specified course, to smooth out the aircraft's ride, and to enhance its responsiveness to pilot commands. In both this case and in the previous example of chemical process control, an important concept, referred to as feedback, plays a major role, as measured signals are fed back and used to adjust the response characteristics of a system.

The examples in the preceding paragraphs represent only a few of an extraordinarily wide variety of applications for the concepts of signals and systems. The importance of these concepts stems not only from the diversity of phenomena and processes in which they arise, but also from the collection of ideas, analytical techniques, and methodologies that have been and are being developed and used to solve problems involving signals and systems. The history of this development extends back over many centuries, and although most of this work was motivated by specific applications, many of these ideas have proven to be of central importance to problems in a far larger variety of contexts than those for which they were originally intended. For example, the tools of Fourier analysis, which form the basis for the frequency-domain analysis of signals and systems, and which we will develop in some detail in this book, can be traced from problems of astronomy studied by the ancient Babylonians to the development of mathematical physics in the eighteenth and nineteenth centuries.

In some of the examples that we have mentioned, the signals vary continuously in time, whereas in others, their evolution is described only at discrete points in time. For example, in the analysis of electrical circuits and mechanical systems we are concerned with signals that vary continuously. On the other hand, the daily closing stock market average is by its very nature a signal that evolves at discrete points in time (i.e., at the close of each day). Rather than a curve as a function of a continuous variable, then, the closing stock market average is a sequence of numbers associated with the discrete time instants at which it is specified. This distinction in the basic description of the evolution of signals and of the systems that respond to or process these signals leads naturally to two parallel frameworks for signal and system analysis, one for phenomena and processes that are described in continuous time and one for those that are described in discrete time.

The concepts and techniques associated both with continuous-time signals and systems and with discrete-time signals and systems have a rich history and are conceptually closely related. Historically, however, because their applications have in the past been sufficiently different, they have for the most part been studied and developed somewhat separately. Continuous-time signals and systems have very strong roots in problems associated with physics and, in the more recent past, with electrical circuits and communications. The techniques of discrete-time signals and systems have strong roots in numerical analysis, statistics, and time-series analysis associated with such applications as the analysis of economic and demographic data. Over the past several decades, however, the disciplines of continuous-time and discrete-time signals and systems have become increasingly entwined and the applications have become highly interrelated. The major impetus for this has come from the dramatic advances in technology for the implementation of systems and for the generation of signals. Specifically, the continuing development of high-speed digital computers, integrated circuits, and sophisticated high-density device fabrication techniques has made it increasingly advantageous to consider processing continuous-time signals by representing them by time samples (i.e., by converting them to discrete-time signals). As one example, the computer control system for a modern high-performance aircraft digitizes sensor outputs such as vehicle speed in order to produce a sequence of sampled measurements which are then processed by the control system.

Because of the growing interrelationship between continuous-time signals and systems and discrete-time signals and systems and because of the close relationship among the concepts and techniques associated with each, we have chosen in this text to develop the concepts of continuous-time and discrete-time signals and systems in parallel. Since many of the concepts are similar (but not identical), by treating them in parallel, insight and intuition can be shared and both the similarities and differences between them become better focused. In addition, as will be evident as we proceed through the material, there are some concepts that are inherently easier to understand in one framework than the other and, once understood, the insight is easily transferable. Furthermore, this parallel treatment greatly facilitates our understanding of the very important practical context in which continuous and discrete time are brought together, namely the sampling of continuous-time signals and the processing of continuous-time signals using discrete-time systems.

As we have so far described them, the notions of signals and systems are extremely general concepts. At this level of generality, however, only the most sweeping statements can be made about the nature of signals and systems, and their properties can be discussed only in the most elementary terms. On the other hand, an important and fundamental notion in dealing with signals and systems is that by carefully choosing subclasses of each with

particular properties that can then be exploited, we can analyze and characterize these signals and systems in great depth. The principal focus in this book is on the particular class of linear time-invariant systems. The properties of linearity and time invariance that define this class lead to a remarkable set of concepts and techniques which are not only of major practical importance but also analytically tractable and intellectually satisfying.

As we have emphasized in this foreword, signal and system analysis has a long history out of which have emerged some basic techniques and fundamental principles which have extremely broad areas of application. Indeed, signal and system analysis is constantly evolving and developing in response to new problems, techniques, and opportunities. We fully expect this development to accelerate in pace as improved technology makes possible the implementation of increasingly complex systems and signal processing techniques. In the future we will see signals and systems tools and concepts applied to an expanding scope of applications. For these reasons, we feel that the topic of signal and system analysis represents a body of knowledge that is of essential concern to the scientist and engineer. We have chosen the set of topics presented in this book, the organization of the presentation, and the problems in each chapter in a way that we feel will most help the reader to obtain a solid foundation in the fundamentals of signal and system analysis; to gain an understanding of some of the very important and basic applications of these fundamentals to problems in filtering, sampling, communications, and feedback system analysis; and to develop some appreciation for an extremely powerful and broadly applicable approach to formulating and solving complex problems.

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SIGNALS AND SYSTEMS
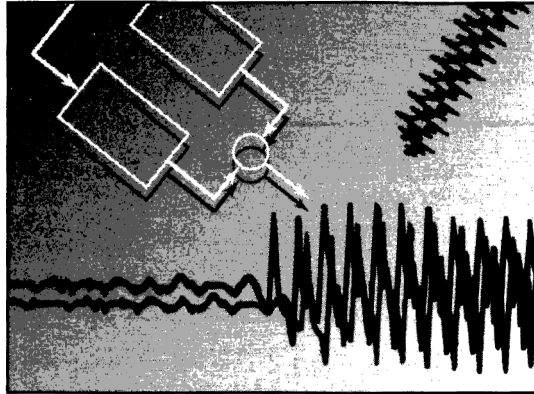
1.0 INTRODUCTION

As described in the Foreword, the intuitive notions of signals and systems arise in a rich variety of contexts. Moreover, as we will see in this book, there is an analytical framework—that is, a language for describing signals and systems and an extremely powerful set of tools for analyzing them—that applies equally well to problems in many fields. In this chapter, we begin our development of the analytical framework for signals and systems by introducing their mathematical description and representations. In the chapters that follow, we build on this foundation in order to develop and describe additional concepts and methods that add considerably both to our understanding of signals and systems and to our ability to analyze and solve problems involving signals and systems that arise in a broad array of applications.

1.1 CONTINUOUS-TIME AND DISCRETE-TIME SIGNALS

1.1.1 Examples and Mathematical Representation

Signals may describe a wide variety of physical phenomena. Although signals can be represented in many ways, in all cases the information in a signal is contained in a pattern of variations of some form. For example, consider the simple circuit in Figure 1.1. In this case, the patterns of variation over time in the source and capacitor voltages, v_s and v_c, are examples of signals. Similarly, as depicted in Figure 1.2, the variations over time of the applied force f and the resulting automobile velocity v are signals. As another example, consider the human vocal mechanism, which produces speech by creating fluctuations in acoustic pressure. Figure 1.3 is an illustration of a recording of such a speech signal, obtained by
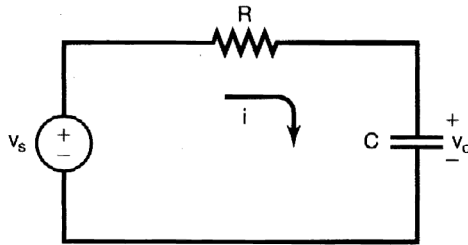
Figure 1.1 A simple *RC* circuit with source voltage v_s and capacitor voltage v_c.
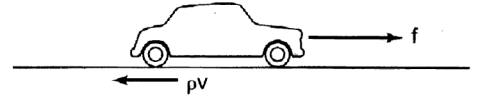
Figure 1.2 An automobile responding to an applied force *f* from the engine and to a retarding frictional force ρv proportional to the automobile's velocity *v*.
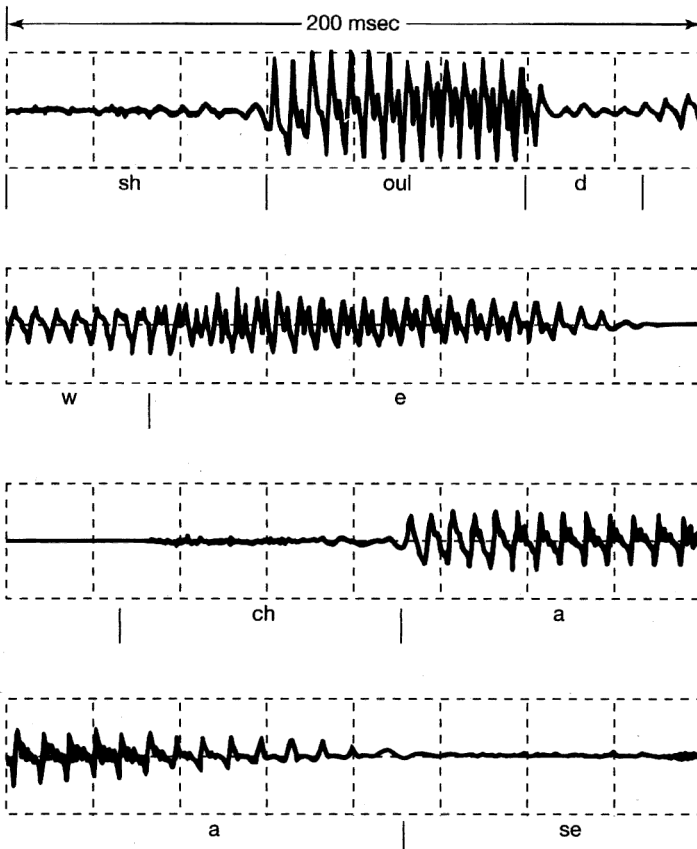
Figure 1.3 Example of a recording of speech. [Adapted from *Applications of Digital Signal Processing*, A.V. Oppenheim, ed. (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1978), p. 121.] The signal represents acoustic pressure variations as a function of time for the spoken words "should we chase." The top line of the figure corresponds to the word "should," the second line to the word "we," and the last two lines to the word "chase." (We have indicated the approximate beginnings and endings of each successive sound in each word.)

using a microphone to sense variations in acoustic pressure, which are then converted into an electrical signal. As can be seen in the figure, different sounds correspond to different patterns in the variations of acoustic pressure, and the human vocal system produces intelligible speech by generating particular sequences of these patterns. Alternatively, for the monochromatic picture, shown in Figure 1.4, it is the pattern of variations in brightness across the image that is important.
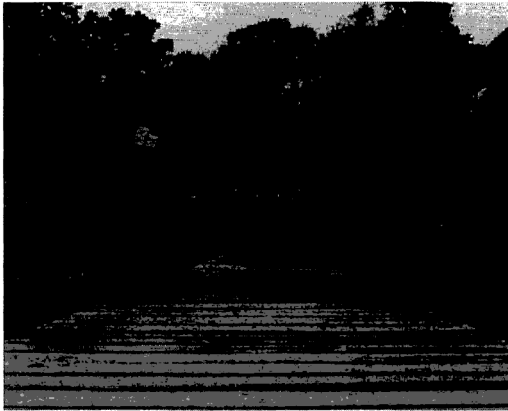
Figure 1.4 A monochromatic picture.

Signals are represented mathematically as functions of one or more independent variables. For example, a speech signal can be represented mathematically by acoustic pressure as a function of time, and a picture can be represented by brightness as a function of two spatial variables. In this book, we focus our attention on signals involving a single independent variable. For convenience, we will generally refer to the independent variable as time, although it may not in fact represent time in specific applications. For example, in geophysics, signals representing variations with depth of physical quantities such as density, porosity, and electrical resistivity are used to study the structure of the earth. Also, knowledge of the variations of air pressure, temperature, and wind speed with altitude are extremely important in meteorological investigations. Figure 1.5 depicts a typical example of annual average vertical wind profile as a function of height. The measured variations of wind speed with height are used in examining weather patterns, as well as wind conditions that may affect an aircraft during final approach and landing.

Throughout this book we will be considering two basic types of signals: continuous-time signals and discrete-time signals. In the case of continuous-time signals the independent variable is continuous, and thus these signals are defined for a continuum of values
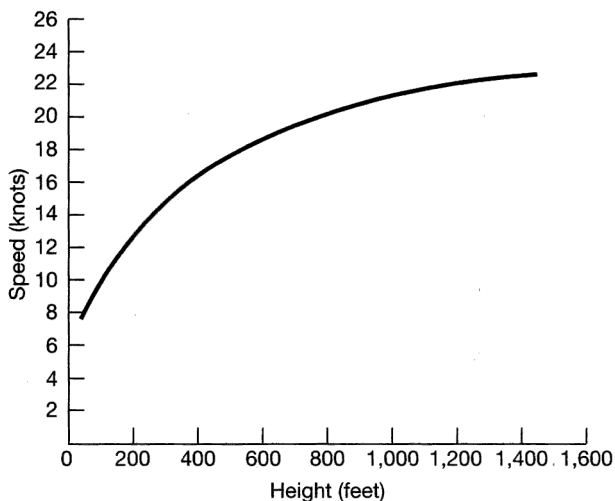
Figure 1.5 Typical annual vertical wind profile. (Adapted from Crawford and Hudson, National Severe Storms Laboratory Report, ESSA ERLTM-NSSL 48, August 1970.)
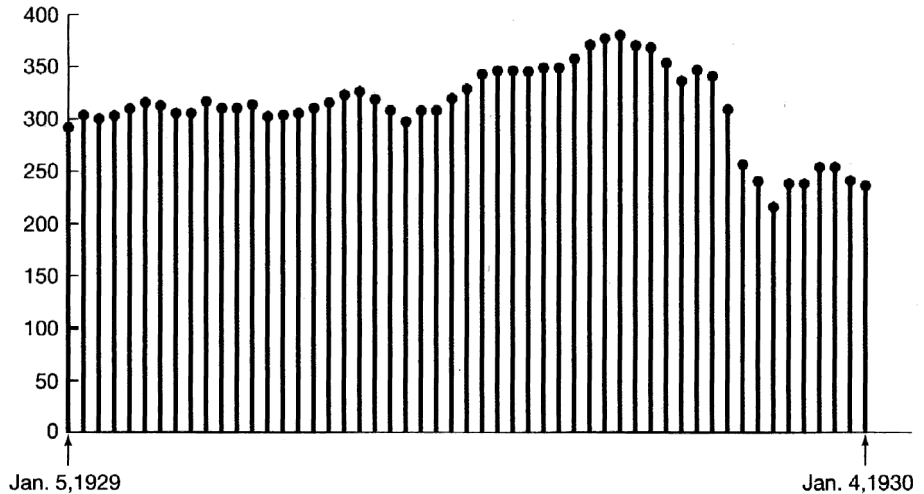
Figure 1.6   An example of a discrete-time signal: The weekly Dow-Jones
stock market index from January 5, 1929, to January 4, 1930.

of the independent variable. On the other hand, discrete-time signals are defined only at
discrete times, and consequently, for these signals, the independent variable takes on only
a discrete set of values. A speech signal as a function of time and atmospheric pressure
as a function of altitude are examples of continuous-time signals. The weekly Dow-Jones
stock market index, as illustrated in Figure 1.6, is an example of a discrete-time signal.
Other examples of discrete-time signals can be found in demographic studies in which
various attributes, such as average budget, crime rate, or pounds of fish caught, are tab-
ulated against such discrete variables as family size, total population, or type of fishing
vessel, respectively.

To distinguish between continuous-time and discrete-time signals, we will use the
symbol t to denote the continuous-time independent variable and n to denote the discrete-
time independent variable. In addition, for continuous-time signals we will enclose the
independent variable in parentheses $(\cdot)$, whereas for discrete-time signals we will use
brackets $[\cdot]$ to enclose the independent variable. We will also have frequent occasions
when it will be useful to represent signals graphically. Illustrations of a continuous-time
signal $x(t)$ and a discrete-time signal $x[n]$ are shown in Figure 1.7. It is important to note
that the discrete-time signal $x[n]$ is defined *only* for integer values of the independent
variable. Our choice of graphical representation for $x[n]$ emphasizes this fact, and for
further emphasis we will on occasion refer to $x[n]$ as a discrete-time *sequence*.

A discrete-time signal $x[n]$ may represent a phenomenon for which the independent
variable is inherently discrete. Signals such as demographic data are examples of this. On
the other hand, a very important class of discrete-time signals arises from the *sampling* of
continuous-time signals. In this case, the discrete-time signal $x[n]$ represents successive
samples of an underlying phenomenon for which the independent variable is continuous.
Because of their speed, computational power, and flexibility, modern digital processors are
used to implement many practical systems, ranging from digital autopilots to digital audio
systems. Such systems require the use of discrete-time sequences representing sampled
versions of continuous-time signals—e.g., aircraft position, velocity, and heading for an

x(t)

0

t

(a)

x[n]

x[0]

x[-1] x[1]

x[2]

-9 -8 -7   -5 -4 -3 -2

-6

10

-1 0 1 2 3 4 5 6 7 7 9

n

(b)
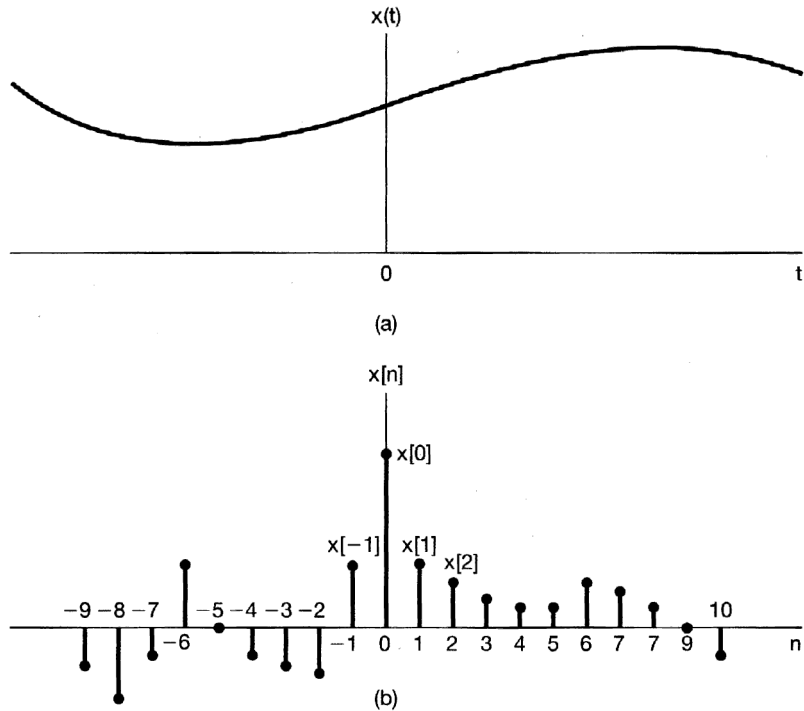
Figure 1.7   Graphical representations of (a) continuous-time and (b) discrete-time signals.

autopilot or speech and music for an audio system. Also, pictures in newspapers—or in this book, for that matter—actually consist of a very fine grid of points, and each of these points represents a sample of the brightness of the corresponding point in the original image. No matter what the source of the data, however, the signal $x[n]$ is defined only for integer values of n. It makes no more sense to refer to the $3\frac{1}{2}$th sample of a digital speech signal than it does to refer to the average budget for a family with $2\frac{1}{2}$ family members.

Throughout most of this book we will treat discrete-time signals and continuous-time signals separately but in parallel, so that we can draw on insights developed in one setting to aid our understanding of another. In Chapter 7 we will return to the question of sampling, and in that context we will bring continuous-time and discrete-time concepts together in order to examine the relationship between a continuous-time signal and a discrete-time signal obtained from it by sampling.

1.1.2 Signal Energy and Power

From the range of examples provided so far, we see that signals may represent a broad variety of phenomena. In many, but not all, applications, the signals we consider are directly related to physical quantities capturing power and energy in a physical system. For example, if $v(t)$ and $i(t)$ are, respectively, the voltage and current across a resistor with resistance R, then the instantaneous power is

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t). \qquad (1.1)$$

The total *energy* expended over the time interval $t_1 \leq t \leq t_2$ is

$$\int_{t_1}^{t_2} p(t)\, dt = \int_{t_1}^{t_2} \frac{1}{R} v^2(t)\, dt, \tag{1.2}$$

and the *average power* over this time interval is

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p(t)\, dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{1}{R} v^2(t)\, dt. \tag{1.3}$$

Similarly, for the automobile depicted in Figure 1.2, the instantaneous power dissipated through friction is $p(t) = bv^2(t)$, and we can then define the total energy and average power over a time interval in the same way as in eqs. (1.2) and (1.3).

With simple physical examples such as these as motivation, it is a common and worthwhile convention to use similar terminology for power and energy for *any* continuous-time signal $x(t)$ or *any* discrete-time signal $x[n]$. Moreover, as we will see shortly, we will frequently find it convenient to consider signals that take on complex values. In this case, the total energy over the time interval $t_1 \leq t \leq t_2$ in a continuous-time signal $x(t)$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 dt, \tag{1.4}$$

where $|x|$ denotes the magnitude of the (possibly complex) number x. The time-averaged power is obtained by dividing eq. (1.4) by the length, $t_2 - t_1$, of the time interval. Similarly, the total energy in a discrete-time signal $x[n]$ over the time interval $n_1 \leq n \leq n_2$ is defined as

$$\sum_{n=n_1}^{n_2} |x[n]|^2, \tag{1.5}$$

and dividing by the number of points in the interval, $n_2 - n_1 + 1$, yields the average power over the interval. It is important to remember that the terms "power" and "energy" are used here independently of whether the quantities in eqs. (1.4) and (1.5) actually are related to physical energy.[1] Nevertheless, we will find it convenient to use these terms in a general fashion.

Furthermore, in many systems we will be interested in examining power and energy in signals over an infinite time interval, i.e., for $-\infty < t < +\infty$ or for $-\infty < n < +\infty$. In these cases, we define the total energy as limits of eqs. (1.4) and (1.5) as the time interval increases without bound. That is, in continuous time,

$$E_\infty \triangleq \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |x(t)|^2\, dt, \tag{1.6}$$

and in discrete time,

$$E_\infty \triangleq \lim_{N \to \infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2. \tag{1.7}$$

---

[1]Even if such a relationship does exist, eqs. (1.4) and (1.5) may have the wrong dimensions and scalings. For example, comparing eqs. (1.2) and (1.4), we see that if $x(t)$ represents the voltage across a resistor, then eq. (1.4) must be divided by the resistance (measured, for example, in ohms) to obtain units of physical energy.

Note that for some signals the integral in eq. (1.6) or sum in eq. (1.7) might not converge—e.g., if $x(t)$ or $x[n]$ equals a nonzero constant value for all time. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

In an analogous fashion, we can define the time-averaged power over an infinite interval as

$$P_\infty \overset{\triangle}{=} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \tag{1.8}$$

and

$$P_\infty \overset{\triangle}{=} \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2 \tag{1.9}$$

in continuous time and discrete time, respectively. With these definitions, we can identify three important classes of signals. The first of these is the class of signals with finite total energy, i.e., those signals for which $E_\infty < \infty$. Such a signal must have zero average power, since in the continuous time case, for example, we see from eq. (1.8) that

$$P_\infty = \lim_{T \to \infty} \frac{E_\infty}{2T} = 0. \tag{1.10}$$

An example of a finite-energy signal is a signal that takes on the value 1 for $0 \leq t \leq 1$ and 0 otherwise. In this case, $E_\infty = 1$ and $P_\infty = 0$.

A second class of signals are those with finite average power P_∞. From what we have just seen, if $P_\infty > 0$, then, of necessity, $E_\infty = \infty$. This, of course, makes sense, since if there is a nonzero average energy per unit time (i.e., nonzero power), then integrating or summing this over an infinite time interval yields an infinite amount of energy. For example, the constant signal $x[n] = 4$ has infinite energy, but average power $P_\infty = 16$. There are also signals for which neither P_∞ nor E_∞ are finite. A simple example is the signal $x(t) = t$. We will encounter other examples of signals in each of these classes in the remainder of this and the following chapters.

1.2 TRANSFORMATIONS OF THE INDEPENDENT VARIABLE

A central concept in signal and system analysis is that of the transformation of a signal. For example, in an aircraft control system, signals corresponding to the actions of the pilot are transformed by electrical and mechanical systems into changes in aircraft thrust or the positions of aircraft control surfaces such as the rudder or ailerons, which in turn are transformed through the dynamics and kinematics of the vehicle into changes in aircraft velocity and heading. Also, in a high-fidelity audio system, an input signal representing music as recorded on a cassette or compact disc is modified in order to enhance desirable characteristics, to remove recording noise, or to balance the several components of the signal (e.g., treble and bass). In this section, we focus on a very limited but important class of elementary signal transformations that involve simple modification of the independent variable, i.e., the time axis. As we will see in this and subsequent sections of this chapter, these elementary transformations allow us to introduce several basic properties of signals and systems. In later chapters, we will find that they also play an important role in defining and characterizing far richer and important classes of systems.

1.2.1 Examples of Transformations of the Independent Variable

A simple and very important example of transforming the independent variable of a signal is a *time shift*. A time shift in discrete time is illustrated in Figure 1.8, in which we have two signals $x[n]$ and $x[n - n_0]$ that are identical in shape, but that are displaced or shifted relative to each other. We will also encounter time shifts in continuous time, as illustrated in Figure 1.9, in which $x(t - t_0)$ represents a delayed (if t_0 is positive) or advanced (if t_0 is negative) version of $x(t)$. Signals that are related in this fashion arise in applications such as radar, sonar, and seismic signal processing, in which several receivers at different locations observe a signal being transmitted through a medium (water, rock, air, etc.). In this case, the difference in propagation time from the point of origin of the transmitted signal to any two receivers results in a time shift between the signals at the two receivers.

A second basic transformation of the time axis is that of *time reversal*. For example, as illustrated in Figure 1.10, the signal $x[-n]$ is obtained from the signal $x[n]$ by a reflection about $n = 0$ (i.e., by reversing the signal). Similarly, as depicted in Figure 1.11, the signal $x(-t)$ is obtained from the signal $x(t)$ by a reflection about $t = 0$. Thus, if $x(t)$ represents an audio tape recording, then $x(-t)$ is the same tape recording played backward. Another transformation is that of *time scaling*. In Figure 1.12 we have illustrated three signals, $x(t)$, $x(2t)$, and $x(t/2)$, that are related by linear scale changes in the independent variable. If we again think of the example of $x(t)$ as a tape recording, then $x(2t)$ is that recording played at twice the speed, and $x(t/2)$ is the recording played at half-speed.

It is often of interest to determine the effect of transforming the independent variable of a given signal $x(t)$ to obtain a signal of the form $x(\alpha t + \beta)$, where α and β are given numbers. Such a transformation of the independent variable preserves the shape of $x(t)$, except that the resulting signal may be linearly stretched if $|\alpha| < 1$, linearly compressed if $|\alpha| > 1$, reversed in time if $\alpha < 0$, and shifted in time if β is nonzero. This is illustrated in the following set of examples.
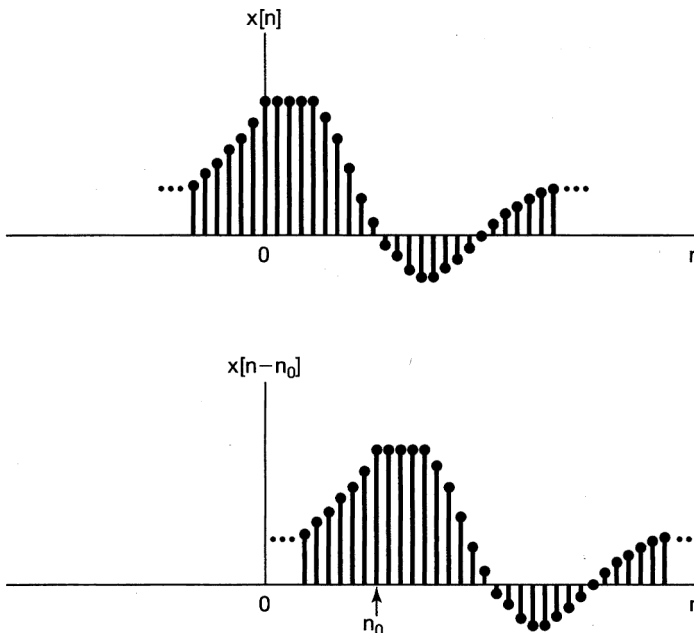
Figure 1.8 Discrete-time signals related by a time shift. In this figure $n_0 > 0$, so that $x[n - n_0]$ is a delayed verson of $x[n]$ (i.e., each point in $x[n]$ occurs later in $x[n - n_0]$).
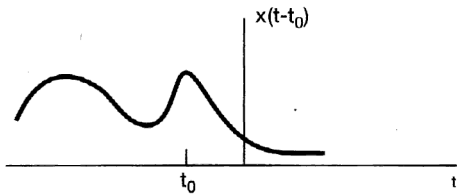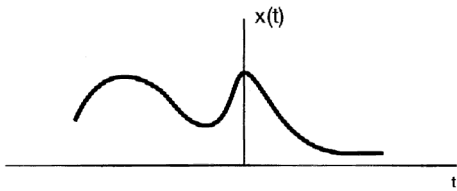
Figure 1.9   Continuous-time signals related by a time shift. In this figure $t_0 < 0$, so that $x(t - t_0)$ is an advanced version of $x(t)$ (i.e., each point in $x(t)$ occurs at an earlier time in $x(t - t_0)$).

Figure 1.10   (a) A discrete-time signal $x[n]$; (b) its reflection $x[-n]$ about $n = 0$.
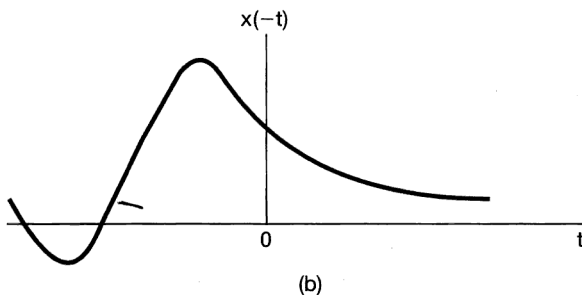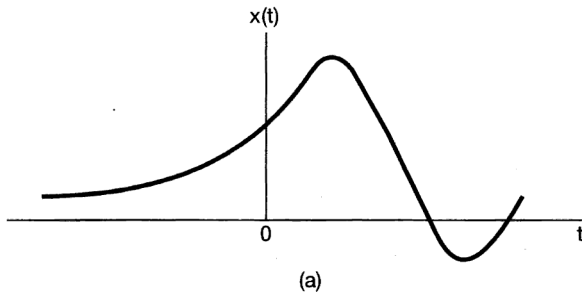
Figure 1.11   (a) A continuous-time signal $x(t)$; (b) its reflection $x(-t)$ about $t = 0$.
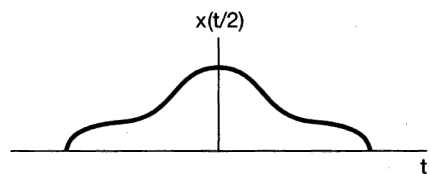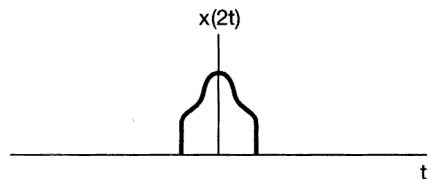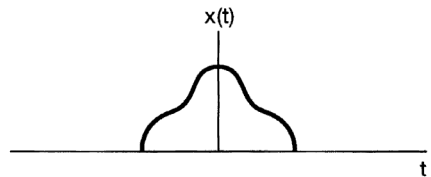
Figure 1.12   Continuous-time signals related by time scaling.

Example 1.1

Given the signal $x(t)$ shown in Figure 1.13(a), the signal $x(t + 1)$ corresponds to an advance (shift to the left) by one unit along the t axis as illustrated in Figure 1.13(b). Specifically, we note that the value of $x(t)$ at $t = t_0$ occurs in $x(t + 1)$ at $t = t_0 - 1$. For
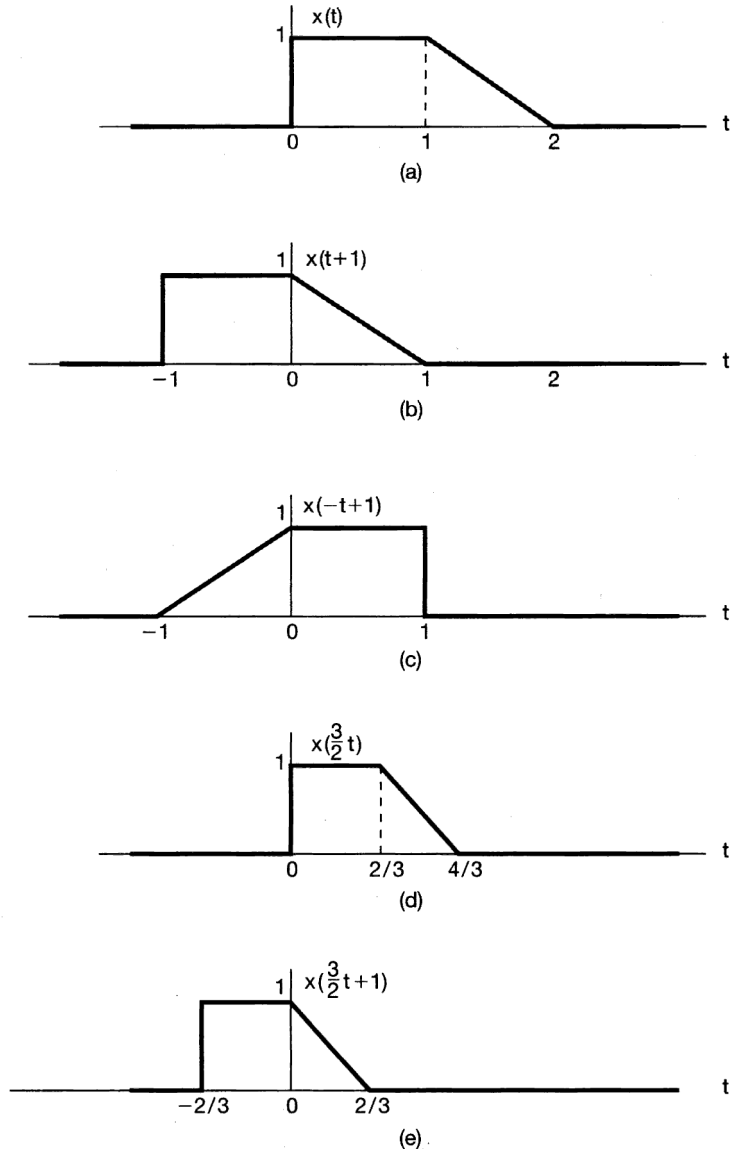
Figure 1.13 (a) The continuous-time signal $x(t)$ used in Examples 1.1–1.3 to illustrate transformations of the independent variable; (b) the time-shifted signal $x(t + 1)$; (c) the signal $x(-t + 1)$ obtained by a time shift and a time reversal; (d) the time-scaled signal $x(\frac{3}{2}t)$; and (e) the signal $x(\frac{3}{2}t + 1)$ obtained by time-shifting and scaling.

example, the value of $x(t)$ at $t = 1$ is found in $x(t + 1)$ at $t = 1 - 1 = 0$. Also, since $x(t)$ is zero for $t < 0$, we have $x(t + 1)$ zero for $t < -1$. Similarly, since $x(t)$ is zero for $t > 2$, $x(t + 1)$ is zero for $t > 1$.

Let us also consider the signal $x(-t + 1)$, which may be obtained by replacing t with $-t$ in $x(t + 1)$. That is, $x(-t + 1)$ is the time reversed version of $x(t + 1)$. Thus, $x(-t + 1)$ may be obtained graphically by reflecting $x(t + 1)$ about the t axis as shown in Figure 1.13(c).

Example 1.2

Given the signal $x(t)$, shown in Figure 1.13(a), the signal $x(\frac{3}{2}t)$ corresponds to a linear compression of $x(t)$ by a factor of $\frac{2}{3}$ as illustrated in Figure 1.13(d). Specifically we note that the value of $x(t)$ at $t = t_0$ occurs in $x(\frac{3}{2}t)$ at $t = \frac{2}{3}t_0$. For example, the value of $x(t)$ at $t = 1$ is found in $x(\frac{3}{2}t)$ at $t = \frac{2}{3}(1) = \frac{2}{3}$. Also, since $x(t)$ is zero for $t < 0$, we have $x(\frac{3}{2}t)$ zero for $t < 0$. Similarly, since $x(t)$ is zero for $t > 2$, $x(\frac{3}{2}t)$ is zero for $t > \frac{4}{3}$.
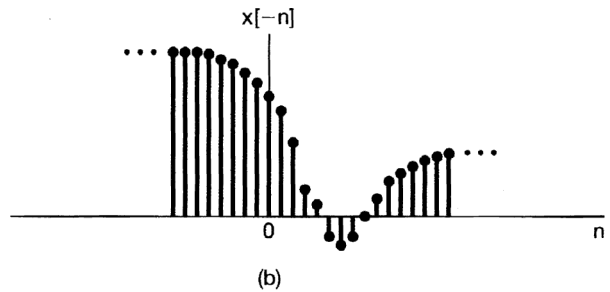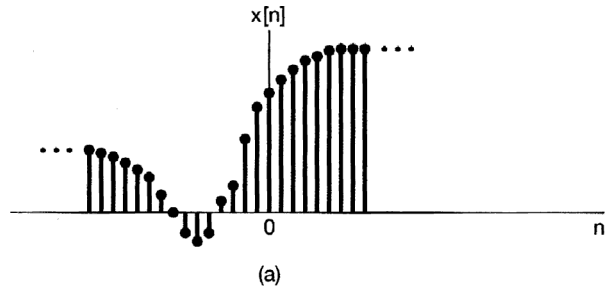
Example 1.3

Suppose that we would like to determine the effect of transforming the independent variable of a given signal, $x(t)$, to obtain a signal of the form $x(\alpha t + \beta)$, where α and β are given numbers. A systematic approach to doing this is to first delay or advance $x(t)$ in accordance with the value of β, and then to perform time scaling and/or time reversal on the resulting signal in accordance with the value of α. The delayed or advanced signal is linearly stretched if $|\alpha| < 1$, linearly compressed if $|\alpha| > 1$, and reversed in time if $\alpha < 0$.

To illustrate this approach, let us show how $x(\frac{3}{2}t + 1)$ may be determined for the signal $x(t)$ shown in Figure 1.13(a). Since $\beta = 1$, we first advance (shift to the left) $x(t)$ by 1 as shown in Figure 1.13(b). Since $|\alpha| = \frac{3}{2}$, we may linearly compress the shifted signal of Figure 1.13(b) by a factor of $\frac{2}{3}$ to obtain the signal shown in Figure 1.13(e).

In addition to their use in representing physical phenomena such as the time shift in a sonar signal and the speeding up or reversal of an audiotape, transformations of the independent variable are extremely useful in signal and system analysis. In Section 1.6 and in Chapter 2, we will use transformations of the independent variable to introduce and analyze the properties of systems. These transformations are also important in defining and examining some important properties of signals.

1.2.2 Periodic Signals

An important class of signals that we will encounter frequently throughout this book is the class of *periodic* signals. A periodic continuous-time signal $x(t)$ has the property that there is a positive value of T for which

$$x(t) = x(t + T) \tag{1.11}$$

for all values of t. In other words, a periodic signal has the property that it is unchanged by a time shift of T. In this case, we say that $x(t)$ is *periodic with period T*. Periodic continuous-time signals arise in a variety of contexts. For example, as illustrated in Problem 2.61, the natural response of systems in which energy is conserved, such as ideal LC circuits without resistive energy dissipation and ideal mechanical systems without frictional losses, are periodic and, in fact, are composed of some of the basic periodic signals that we will introduce in Section 1.3.
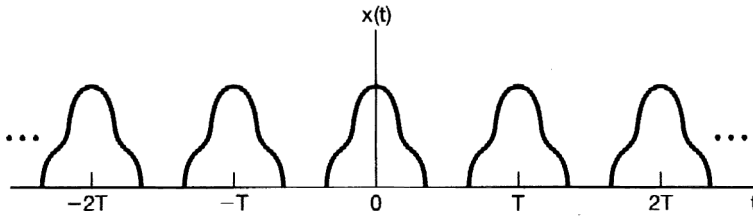
Figure 1.14   A continuous-time periodic signal.

An example of a periodic continuous-time signal is given in Figure 1.14. From the figure or from eq. (1.11), we can readily deduce that if $x(t)$ is periodic with period T, then $x(t) = x(t + mT)$ for all t and for any integer m. Thus, $x(t)$ is also periodic with period $2T, 3T, 4T, \ldots$. The *fundamental period* T_0 of $x(t)$ is the smallest positive value of T for which eq. (1.11) holds. This definition of the fundamental period works, except if $x(t)$ is a constant. In this case the fundamental period is undefined, since $x(t)$ is periodic for *any* choice of T (so there is no smallest positive value). A signal $x(t)$ that is not periodic will be referred to as an *aperiodic* signal.

Periodic signals are defined analogously in discrete time. Specifically, a discrete-time signal $x[n]$ is periodic with period N, where N is a positive integer, if it is unchanged by a time shift of N, i.e., if

$$x[n] = x[n + N] \tag{1.12}$$

for all values of n. If eq. (1.12) holds, then $x[n]$ is also periodic with period $2N, 3N, \ldots$. The *fundamental period* N_0 is the smallest positive value of N for which eq. (1.12) holds. An example of a discrete-time periodic signal with fundamental period $N_0 = 3$ is shown in Figure 1.15.
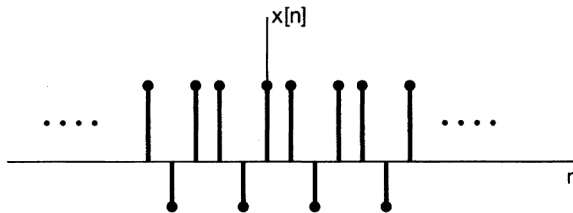
Figure 1.15   A discrete-time periodic signal with fundamental period $N_0 = 3$.

Example 1.4

Let us illustrate the type of problem solving that may be required in determining whether or not a given signal is periodic. The signal whose periodicity we wish to check is given by

$$x(t) = \begin{cases} \cos(t) & \text{if } t < 0 \\ \sin(t) & \text{if } t \geq 0 \end{cases}. \tag{1.13}$$

From trigonometry, we know that $\cos(t + 2\pi) = \cos(t)$ and $\sin(t + 2\pi) = \sin(t)$. Thus, considering $t > 0$ and $t < 0$ separately, we see that $x(t)$ does repeat itself over every interval of length 2π. However, as illustrated in Figure 1.16, $x(t)$ also has a discontinuity at the time origin that does not recur at any other time. Since every feature in the shape of a periodic signal *must* recur periodically, we conclude that the signal $x(t)$ is not periodic.
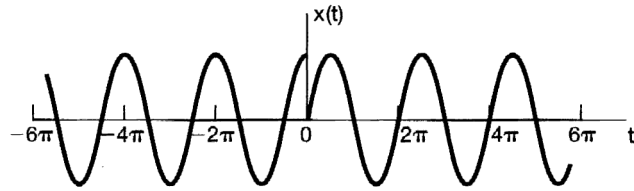
Figure 1.16   The signal $x(t)$ considered in Example 1.4.

1.2.3 Even and Odd Signals

Another set of useful properties of signals relates to their symmetry under time reversal. A signal $x(t)$ or $x[n]$ is referred to as an *even* signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin. In continuous time a signal is even if

$$x(-t) = x(t), \tag{1.14}$$

while a discrete-time signal is even if

$$x[-n] = x[n]. \tag{1.15}$$

A signal is referred to as *odd* if

$$x(-t) = -x(t), \tag{1.16}$$
$$x[-n] = -x[n]. \tag{1.17}$$

An odd signal must necessarily be 0 at $t = 0$ or $n = 0$, since eqs. (1.16) and (1.17) require that $x(0) = -x(0)$ and $x[0] = -x[0]$. Examples of even and odd continuous-time signals are shown in Figure 1.17.
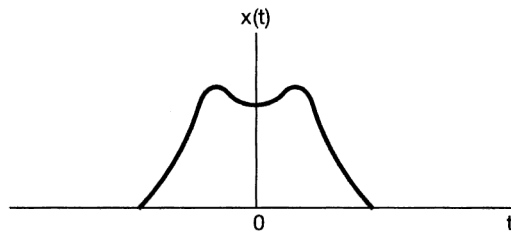
(a)
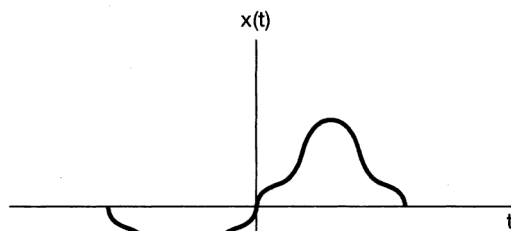
(b)

Figure 1.17   (a) An even continuous-time signal; (b) an odd continuous-time signal.

$$x[n] = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases}$$

$$\mathcal{E}v\{x[n]\} = \begin{cases} \frac{1}{2}, n < 0 \\ 1, n = 0 \\ \frac{1}{2}, n > 0 \end{cases}$$

$$\mathcal{O}d\{x[n]\} = \begin{cases} -\frac{1}{2}, n < 0 \\ 0, n = 0 \\ \frac{1}{2}, n > 0 \end{cases}$$
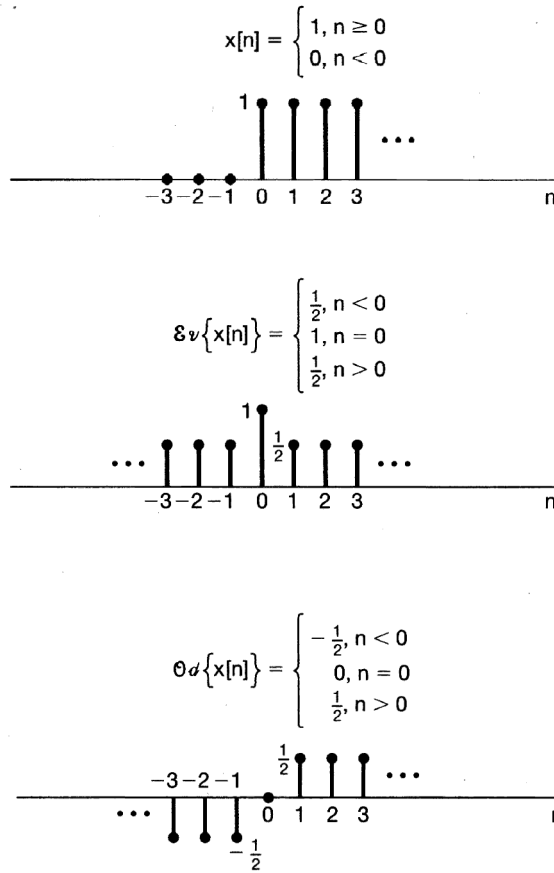
Figure 1.18 Example of the even-odd decomposition of a discrete-time signal.

An important fact is that any signal can be broken into a sum of two signals, one of which is even and one of which is odd. To see this, consider the signal

$$\mathcal{E}v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)], \tag{1.18}$$

which is referred to as the *even part* of $x(t)$. Similarly, the *odd part* of $x(t)$ is given by

$$\mathcal{O}d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]. \tag{1.19}$$

It is a simple exercise to check that the even part is in fact even, that the odd part is odd, and that $x(t)$ is the sum of the two. Exactly analogous definitions hold in the discrete-time case. An example of the even-odd decomposition of a discrete-time signal is given in Figure 1.18.

1.3 EXPONENTIAL AND SINUSOIDAL SIGNALS

In this section and the next, we introduce several basic continuous-time and discrete-time signals. Not only do these signals occur frequently, but they also serve as basic building blocks from which we can construct many other signals.

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

The continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}, \qquad (1.20)$$

where C and a are, in general, complex numbers. Depending upon the values of these parameters, the complex exponential can exhibit several different characteristics.

Real Exponential Signals

As illustrated in Figure 1.19, if C and a are real [in which case $x(t)$ is called a *real exponential*], there are basically two types of behavior. If a is positive, then as t increases $x(t)$ is a growing exponential, a form that is used in describing many different physical processes, including chain reactions in atomic explosions and complex chemical reactions. If a is negative, then $x(t)$ is a decaying exponential, a signal that is also used to describe a wide variety of phenomena, including the process of radioactive decay and the responses of RC circuits and damped mechanical systems. In particular, as shown in Problems 2.61 and 2.62, the natural responses of the circuit in Figure 1.1 and the automobile in Figure 1.2 are decaying exponentials. Also, we note that for $a = 0$, $x(t)$ is constant.
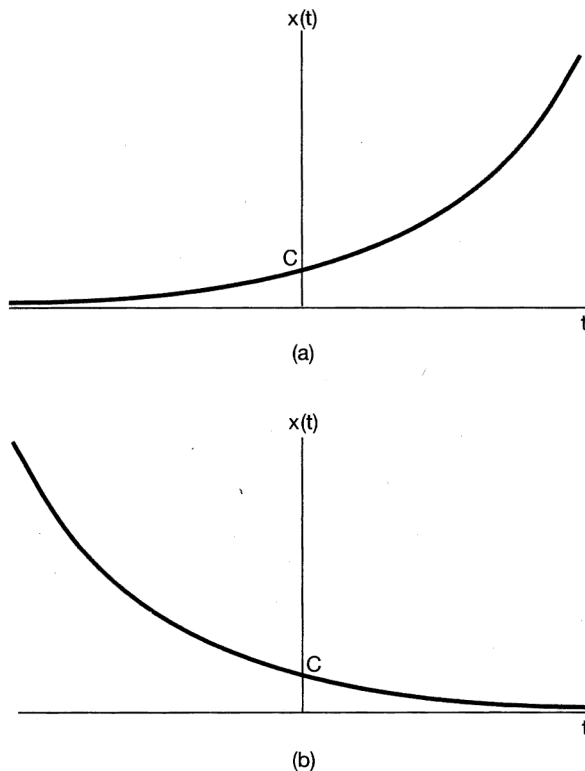
Figure 1.19 Continuous-time real exponential $x(t) = Ce^{at}$: (a) $a > 0$; (b) $a < 0$.

Periodic Complex Exponential and Sinusoidal Signals

A second important class of complex exponentials is obtained by constraining a to be purely imaginary. Specifically, consider

$$x(t) = e^{j\omega_0 t}. \tag{1.21}$$

An important property of this signal is that it is periodic. To verify this, we recall from eq. (1.11) that $x(t)$ will be periodic with period T if

$$e^{j\omega_0 t} = e^{j\omega_0(t+T)}. \tag{1.22}$$

Or, since

$$e^{j\omega_0(t+T)} = e^{j\omega_0 t} e^{j\omega_0 T},$$

it follows that for periodicity, we must have

$$e^{j\omega_0 T} = 1. \tag{1.23}$$

If $\omega_0 = 0$, then $x(t) = 1$, which is periodic for any value of T. If $\omega_0 \neq 0$, then the fundamental period T_0 of $x(t)$—that is, the smallest positive value of T for which eq. (1.23) holds—is

$$T_0 = \frac{2\pi}{|\omega_0|}. \tag{1.24}$$

Thus, the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period.

A signal closely related to the periodic complex exponential is the *sinusoidal signal*

$$x(t) = A\cos(\omega_0 t + \phi), \tag{1.25}$$

as illustrated in Figure 1.20. With seconds as the units of t, the units of ϕ and ω_0 are radians and radians per second, respectively. It is also common to write $\omega_0 = 2\pi f_0$, where f_0 has the units of cycles per second, or hertz (Hz). Like the complex exponential signal, the sinusoidal signal is periodic with fundamental period T_0 given by eq. (1.24). Sinusoidal and
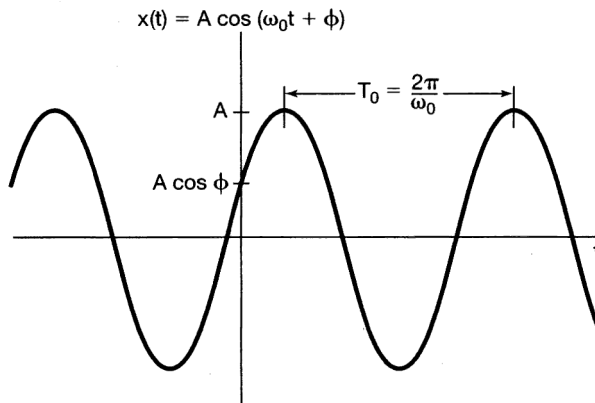
Figure 1.20 Continuous-time sinusoidal signal.

complex exponential signals are also used to describe the characteristics of many physical processes—in particular, physical systems in which energy is conserved. For example, as shown in Problem 2.61, the natural response of an *LC* circuit is sinusoidal, as is the simple harmonic motion of a mechanical system consisting of a mass connected by a spring to a stationary support. The acoustic pressure variations corresponding to a single musical tone are also sinusoidal.

By using Euler's relation,[2] the complex exponential in eq. (1.21) can be written in terms of sinusoidal signals with the same fundamental period:

$$e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t. \tag{1.26}$$

Similarly, the sinusoidal signal of eq. (1.25) can be written in terms of periodic complex exponentials, again with the same fundamental period:

$$A \cos(\omega_0 t + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 t} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 t}. \tag{1.27}$$

Note that the two exponentials in eq. (1.27) have complex amplitudes. Alternatively, we can express a sinusoid in terms of a complex exponential signal as

$$A \cos(\omega_0 t + \phi) = A \mathcal{R}e\{e^{j(\omega_0 t + \phi)}\}, \tag{1.28}$$

where, if c is a complex number, $\mathcal{R}e\{c\}$ denotes its real part. We will also use the notation $\mathcal{I}m\{c\}$ for the imaginary part of c, so that, for example,

$$A \sin(\omega_0 t + \phi) = A \mathcal{I}m\{e^{j(\omega_0 t + \phi)}\}. \tag{1.29}$$

From eq. (1.24), we see that the fundamental period T_0 of a continuous-time sinusoidal signal or a periodic complex exponential is inversely proportional to $|\omega_0|$, which we will refer to as the *fundamental frequency*. From Figure 1.21, we see graphically what this means. If we decrease the magnitude of ω_0, we slow down the rate of oscillation and therefore increase the period. Exactly the opposite effects occur if we increase the magnitude of ω_0. Consider now the case $\omega_0 = 0$. In this case, as we mentioned earlier, $x(t)$ is constant and therefore is periodic with period T for any positive value of T. Thus, the fundamental period of a constant signal is undefined. On the other hand, there is no ambiguity in defining the fundamental frequency of a constant signal to be zero. That is, a constant signal has a zero rate of oscillation.

Periodic signals—and in particular, the complex periodic exponential signal in eq. (1.21) and the sinusoidal signal in eq. (1.25)—provide important examples of signals with infinite total energy but finite average power. For example, consider the periodic exponential signal of eq. (1.21), and suppose that we calculate the total energy and average power in this signal over one period:

$$\begin{aligned} E_{\text{period}} &= \int_0^{T_0} |e^{j\omega_0 t}|^2 \, dt \\ &= \int_0^{T_0} 1 \cdot dt = T_0, \end{aligned} \tag{1.30}$$

---

[2]Euler's relation and other basic ideas related to the manipulation of complex numbers and exponentials are considered in the mathematical review section of the problems at the end of the chapter.

$x_1(t) = \cos \omega_1 t$

(a)

$x_2(t) = \cos \omega_2 t$

(b)

$x_3(t) = \cos \omega_3 t$

(c)
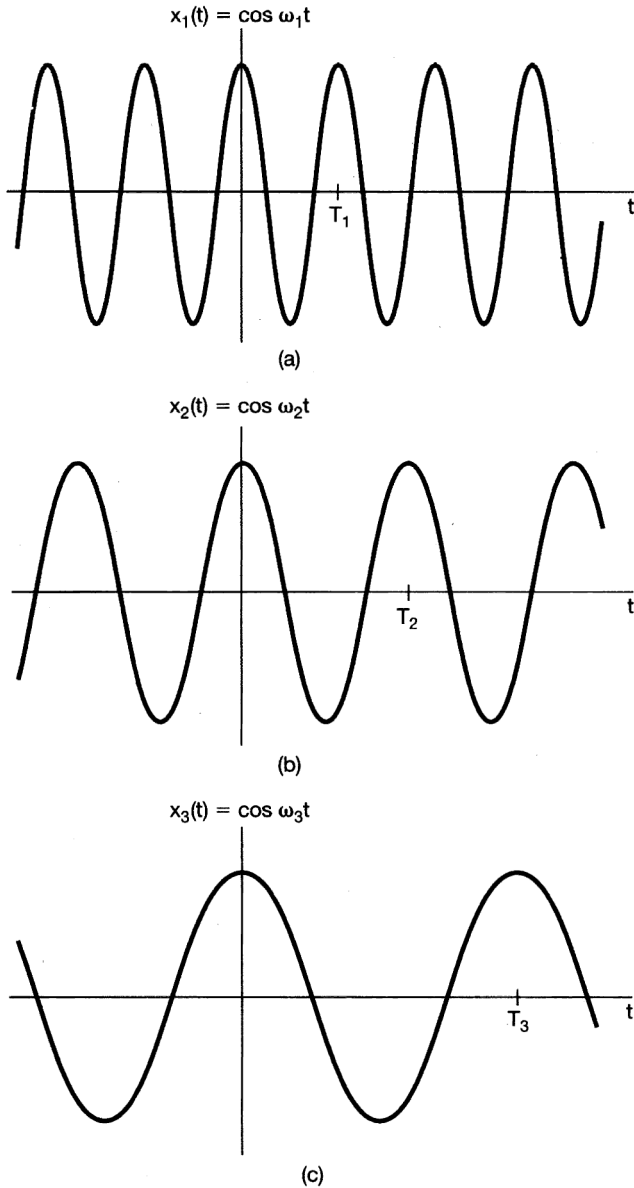
Figure 1.21 Relationship between the fundamental frequency and period for continuous-time sinusoidal signals; here, $\omega_1 > \omega_2 > \omega_3$, which implies that $T_1 < T_2 < T_3$.

$$P_{\text{period}} = \frac{1}{T_0} E_{\text{period}} = 1. \tag{1.31}$$

Since there are an infinite number of periods as t ranges from $-\infty$ to $+\infty$, the total energy integrated over all time is infinite. However, each period of the signal looks exactly the same. Since the average power of the signal equals 1 over each period, averaging over multiple periods always yields an average power of 1. That is, the complex periodic ex-

ponential signal has finite average power equal to

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |e^{j\omega_0 t}|^2 \, dt = 1. \tag{1.32}$$

Problem 1.3 provides additional examples of energy and power calculations for periodic and aperiodic signals.

Periodic complex exponentials will play a central role in much of our treatment of signals and systems, in part because they serve as extremely useful building blocks for many other signals. We will often find it useful to consider sets of *harmonically related* complex exponentials—that is, sets of periodic exponentials, all of which are periodic with a common period T_0. Specifically, a necessary condition for a complex exponential $e^{j\omega t}$ to be periodic with period T_0 is that

$$e^{j\omega T_0} = 1, \tag{1.33}$$

which implies that ωT_0 is a multiple of 2π, i.e.,

$$\omega T_0 = 2\pi k, \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{1.34}$$

Thus, if we define

$$\omega_0 = \frac{2\pi}{T_0}, \tag{1.35}$$

we see that, to satisfy eq. (1.34), ω must be an integer multiple of ω_0. That is, a harmonically related set of complex exponentials is a set of periodic exponentials with fundamental frequencies that are all multiples of a single positive frequency ω_0:

$$\phi_k(t) = e^{jk\omega_0 t}, \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{1.36}$$

For $k = 0$, $\phi_k(t)$ is a constant, while for any other value of k, $\phi_k(t)$ is periodic with fundamental frequency $|k|\omega_0$ and fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}. \tag{1.37}$$

The kth harmonic $\phi_k(t)$ is still periodic with period T_0 as well, as it goes through exactly $|k|$ of its fundamental periods during any time interval of length T_0.

Our use of the term "harmonic" is consistent with its use in music, where it refers to tones resulting from variations in acoustic pressure at frequencies that are integer multiples of a fundamental frequency. For example, the pattern of vibrations of a string on an instrument such as a violin can be described as a superposition—i.e., a weighted sum—of harmonically related periodic exponentials. In Chapter 3, we will see that we can build a very rich class of periodic signals using the harmonically related signals of eq. (1.36) as the building blocks.

Example 1.5

It is sometimes desirable to express the sum of two complex exponentials as the product of a single complex exponential and a single sinusoid. For example, suppose we wish to

plot the magnitude of the signal

$$x(t) = e^{j2t} + e^{j3t}. \tag{1.38}$$

To do this, we first factor out a complex exponential from the right side of eq. (1.38), where the frequency of this exponential factor is taken as the average of the frequencies of the two exponentials in the sum. Doing this, we obtain

$$x(t) = e^{j2.5t}(e^{-j0.5} + e^{j0.5t}), \tag{1.39}$$

which, because of Euler's relation, can be rewritten as

$$x(t) = 2e^{j2.5t}\cos(0.5t). \tag{1.40}$$

From this, we can directly obtain an expression for the magnitude of $x(t)$:

$$|x(t)| = 2|\cos(0.5t)|. \tag{1.41}$$

Here, we have used the fact that the magnitude of the complex exponential $e^{j2.5t}$ is always unity. Thus, $|x(t)|$ is what is commonly referred to as a full-wave rectified sinusoid, as shown in Figure 1.22.
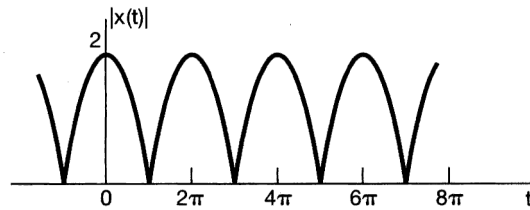
Figure 1.22   The full-wave rectified sinusoid of Example 1.5.

General Complex Exponential Signals

The most general case of a complex exponential can be expressed and interpreted in terms of the two cases we have examined so far: the real exponential and the periodic complex exponential. Specifically, consider a complex exponential Ce^{at}, where C is expressed in polar form and a in rectangular form. That is,

$$C = |C|e^{j\theta}$$

and

$$a = r + j\omega_0.$$

Then

$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t + \theta)}. \tag{1.42}$$

Using Euler's relation, we can expand this further as

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta). \tag{1.43}$$

Thus, for $r = 0$, the real and imaginary parts of a complex exponential are sinusoidal. For $r > 0$ they correspond to sinusoidal signals multiplied by a growing exponential, and for $r < 0$ they correspond to si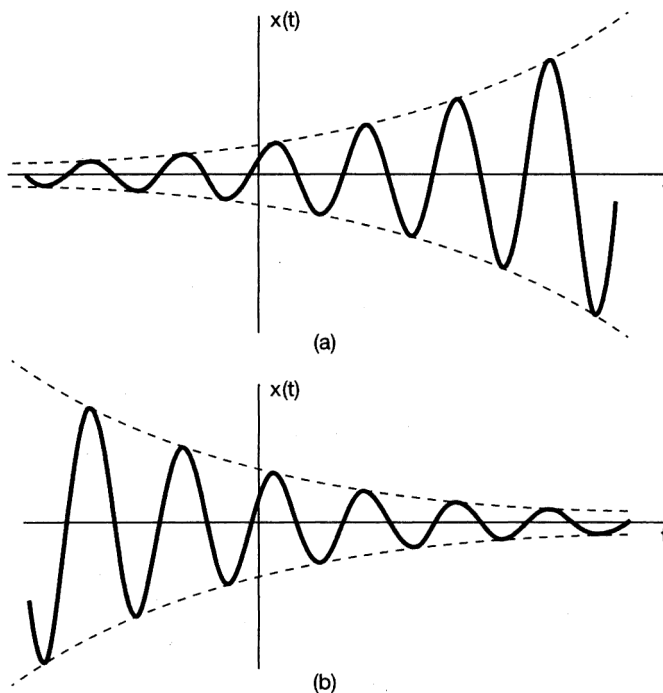nusoidal signals multiplied by a decaying exponential. These two cases are shown in Figure 1.23. The dashed lines in the figure correspond to the functions $\pm|C|e^{rt}$. From eq. (1.42), we see that $|C|e^{rt}$ is the magnitude of the complex exponential. Thus, the dashed curves act as an envelope for the oscillatory curve in the figure in that the peaks of the oscillations just reach these curves, and in this way the envelope provides us with a convenient way to visualize the general trend in the amplitude of the oscillations.

Figure 1.23   (a) Growing sinusoidal signal $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r > 0$; (b) decaying sinusoid $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r < 0$.

Sinusoidal signals multiplied by decaying exponentials are commonly referred to as *damped sinusoids*. Examples of damped sinusoids arise in the response of *RLC* circuits and in mechanical systems containing both damping and restoring forces, such as automotive suspension systems. These kinds of systems have mechanisms that dissipate energy (resistors, damping forces such as friction) with oscillations that decay in time. Examples illustrating such systems and their damped sinusoidal natural responses can be found in Problems 2.61 and 2.62.

1.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals

As in continuous time, an important signal in discrete time is the *complex exponential signal* or *sequence,* defined by

$$x[n] = C\alpha^n, \tag{1.44}$$

where C and α are, in general, complex numbers. This could alternatively be expressed in the form

$$x[n] = Ce^{\beta n}, \tag{1.45}$$

where

$$\alpha = e^{\beta}.$$

Although the form of the discrete-time complex exponential sequence given in eq. (1.45) is more analogous to the form of the continuous-time exponential, it is often more convenient to express the discrete-time complex exponential sequence in the form of eq. (1.44).

Real Exponential Signals

If C and α are real, we can have one of several types of behavior, as illustrated in Figure 1.24. If $|\alpha| > 1$ the magnitude of the signal grows exponentially with n, while if $|\alpha| < 1$ we have a decaying exponential. Furthermore, if α is positive, all the values of $C\alpha^n$ are of the same sign, but if α is negative then the sign of $x[n]$ alternates. Note also that if $\alpha = 1$ then $x[n]$ is a constant, whereas if $\alpha = -1$, $x[n]$ alternates in value between $+C$ and $-C$. Real-valued discrete-time exponentials are often used to describe population growth as a function of generation and total return on investment as a function of day, month, or quarter.

Sinusoidal Signals

Another important complex exponential is obtained by using the form given in eq. (1.45) and by constraining β to be purely imaginary (so that $|\alpha| = 1$). Specifically, consider

$$x[n] = e^{j\omega_0 n}. \tag{1.46}$$

As in the continuous-time case, this signal is closely related to the sinusoidal signal

$$x[n] = A\cos(\omega_0 n + \phi). \tag{1.47}$$

If we take n to be dimensionless, then both ω_0 and ϕ have units of radians. Three examples of sinusoidal sequences are shown in Figure 1.25.

As before, Euler's relation allows us to relate complex exponentials and sinusoids:

$$e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n \tag{1.48}$$

and

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}. \tag{1.49}$$

The signals in eqs. (1.46) and (1.47) are examples of discrete-time signals with infinite total energy but finite average power. For example, since $|e^{j\omega_0 n}|^2 = 1$, every sample of the signal in eq. (1.46) contributes 1 to the signal's energy. Thus, the total energy for $-\infty < n < \infty$ is infinite, while the average power per time point is obviously equal to 1. Other examples of energy and power calculations for discrete-time signals are given in Problem 1.3.
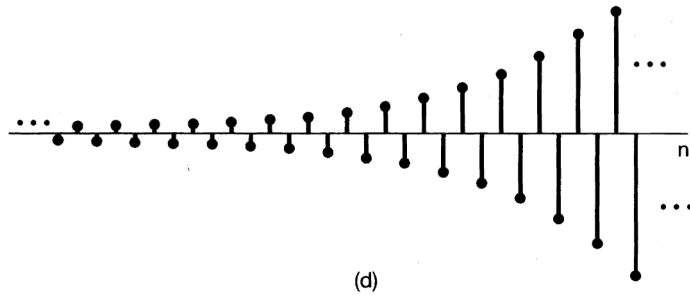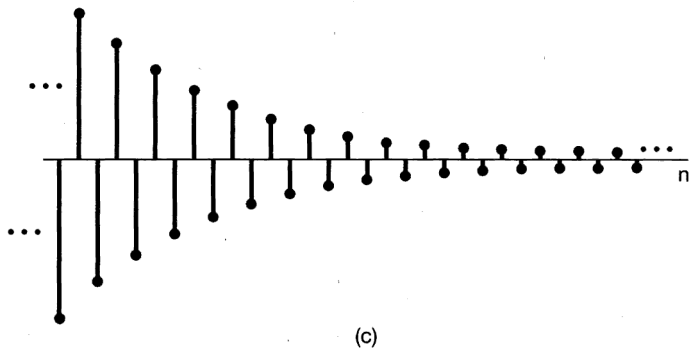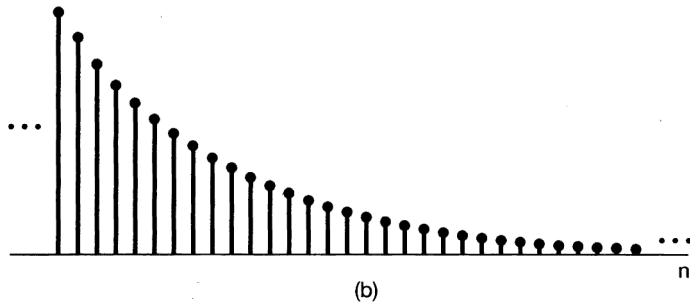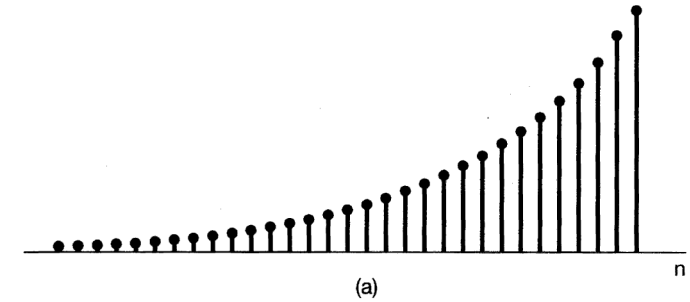
Figure 1.24   The real exponential signal $x[n] = C\alpha^n$: (a) $\alpha > 1$; (b) $0 < \alpha < 1$; (c) $-1 < \alpha < 0$; (d) $\alpha < -1$.

$$x[n] = \cos(2\pi n/12)$$

(a)

$$x[n] = \cos(8\pi n/31)$$

(b)

$$x[n] = \cos(n/6)$$

(c)
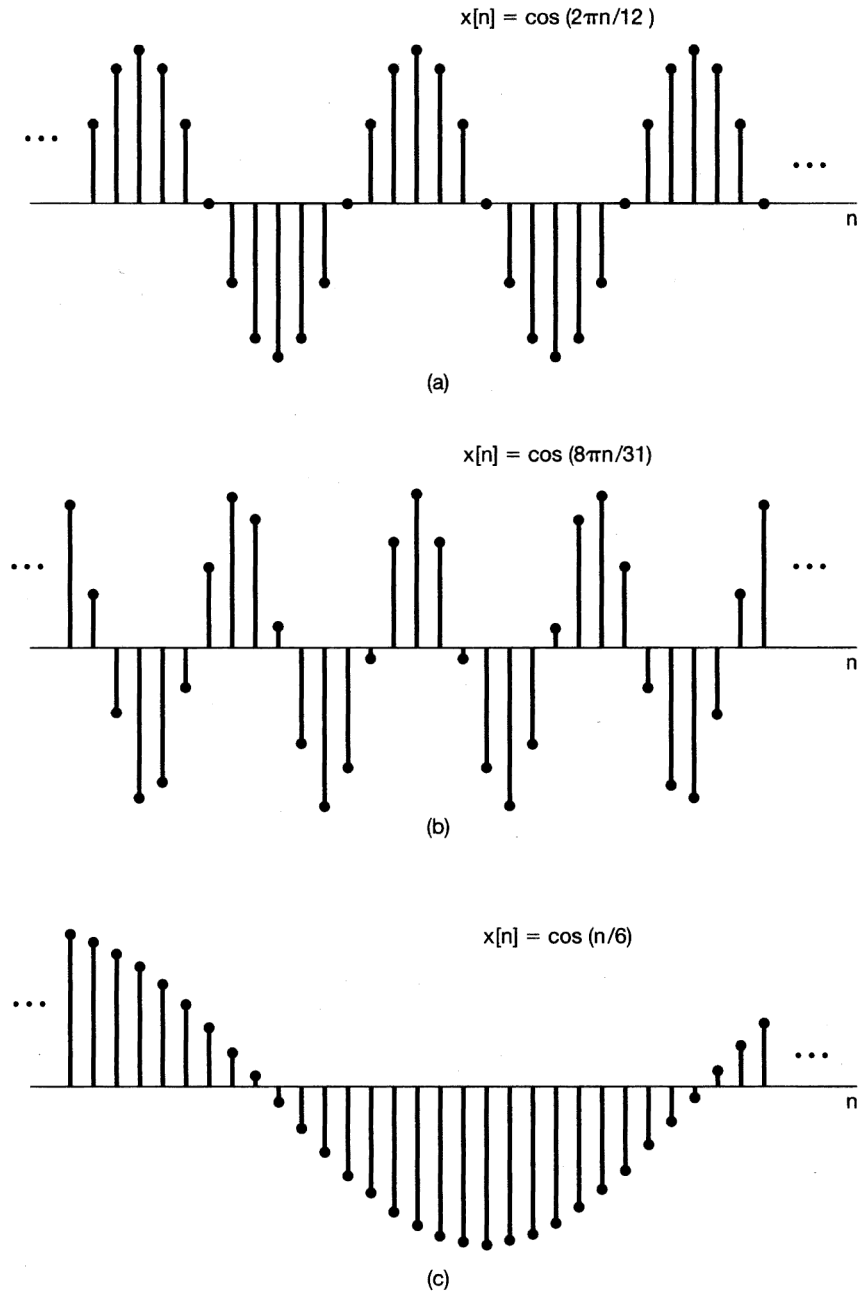
Figure 1.25   Discrete-time sinusoidal signals.

General Complex Exponential Signals

The general discrete-time complex exponential can be written and interpreted in terms of real exponentials and sinusoidal signals. Specifically, if we write C and α in polar form,
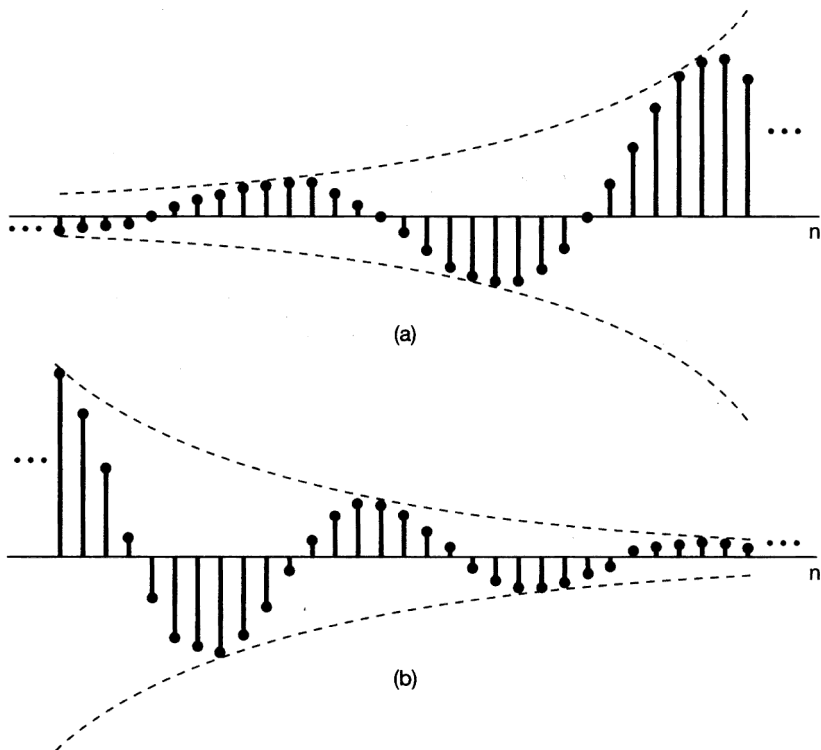
viz.,

$$C = |C|e^{j\theta}$$

and

$$\alpha = |\alpha|e^{j\omega_0},$$

then

$$C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta). \qquad (1.50)$$

Thus, for $|\alpha| = 1$, the real and imaginary parts of a complex exponential sequence are sinusoidal. For $|\alpha| < 1$ they correspond to sinusoidal sequences multiplied by a decaying exponential, while for $|\alpha| > 1$ they correspond to sinusoidal sequences multiplied by a growing exponential. Examples of these signals are depicted in Figure 1.26.

Figure 1.26   (a) Growing discrete-time sinusoidal signals; (b) decaying discrete-time sinusoid.

1.3.3 Periodicity Properties of Discrete-Time Complex Exponentials

While there are many similarities between continuous-time and discrete-time signals, there are also a number of important differences. One of these concerns the discrete-time exponential signal $e^{j\omega_0 n}$. In Section 1.3.1, we identified the following two properties of its

continuous-time counterpart $e^{j\omega_0 t}$: (1) the larger the magnitude of ω_0, the higher is the rate of oscillation in the signal; and (2) $e^{j\omega_0 t}$ is periodic for any value of ω_0. In this section we describe the discrete-time versions of both of these properties, and as we will see, there are definite differences between each of these and its continuous-time counterpart.

The fact that the first of these properties is different in discrete time is a direct consequence of another extremely important distinction between discrete-time and continuous-time complex exponentials. Specifically, consider the discrete-time complex exponential with frequency $\omega_0 + 2\pi$:

$$e^{j(\omega_0 + 2\pi)n} = e^{j2\pi n} e^{j\omega_0 n} = e^{j\omega_0 n}. \tag{1.51}$$

From eq. (1.51), we see that the exponential at frequency $\omega_0 + 2\pi$ is the *same* as that at frequency ω_0. Thus, we have a very different situation from the continuous-time case, in which the signals $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0. In discrete time, these signals are not distinct, as the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on. Therefore, in considering discrete-time complex exponentials, we need only consider a frequency interval of length 2π in which to choose ω_0. Although, according to eq. (1.51), any interval of length 2π will do, on most occasions we will use the interval $0 \leq \omega_0 < 2\pi$ or the interval $-\pi \leq \omega_0 < \pi$.

Because of the periodicity implied by eq. (1.51), the signal $e^{j\omega_0 n}$ does *not* have a continually increasing rate of oscillation as ω_0 is increased in magnitude. Rather, as illustrated in Figure 1.27, as we increase ω_0 from 0, we obtain signals that oscillate more and more rapidly until we reach $\omega_0 = \pi$. As we continue to increase ω_0, we *decrease* the rate of oscillation until we reach $\omega_0 = 2\pi$, which produces the same constant sequence as $\omega_0 = 0$. Therefore, the low-frequency (that is, slowly varying) discrete-time exponentials have values of ω_0 near 0, 2π, and any other even multiple of π, while the high frequencies (corresponding to rapid variations) are located near $\omega_0 = \pm\pi$ and other odd multiples of π. Note in particular that for $\omega_0 = \pi$ or any other odd multiple of π,

$$e^{j\pi n} = (e^{j\pi})^n = (-1)^n, \tag{1.52}$$

so that this signal oscillates rapidly, changing sign at each point in time [as illustrated in Figure 1.27(e)].

The second property we wish to consider concerns the periodicity of the discrete-time complex exponential. In order for the signal $e^{j\omega_0 n}$ to be periodic with period $N > 0$, we must have

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}, \tag{1.53}$$

or equivalently,

$$e^{j\omega_0 N} = 1. \tag{1.54}$$

For eq. (1.54) to hold, $\omega_0 N$ must be a multiple of 2π. That is, there must be an integer m such that

$$\omega_0 N = 2\pi m, \tag{1.55}$$

or equivalently,

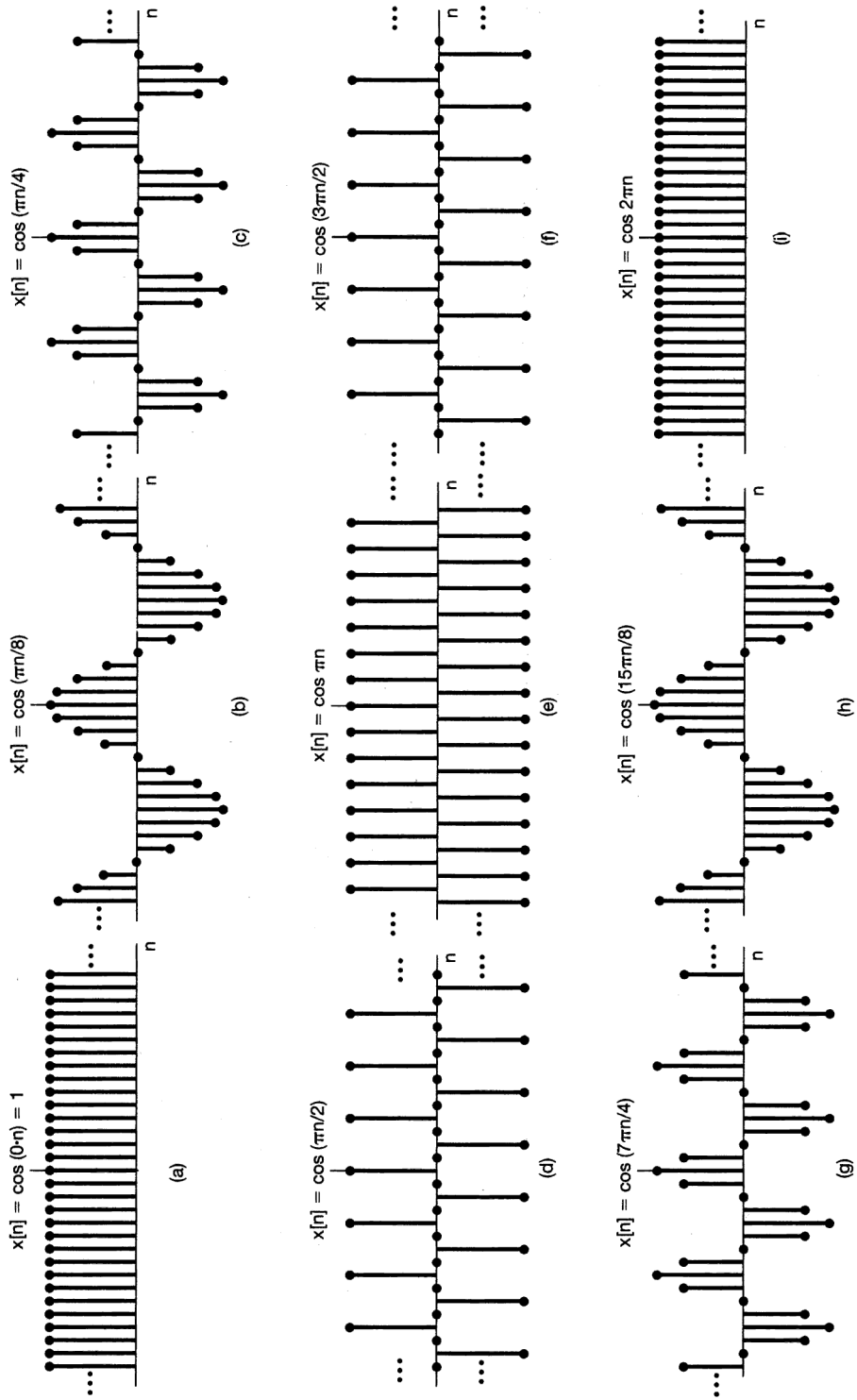$$\frac{\omega_0}{2\pi} = \frac{m}{N}. \tag{1.56}$$

Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

27

According to eq. (1.56), the signal $e^{j\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number and is not periodic otherwise. These same observations also hold for discrete-time sinusoids. For example, the signals depicted in Figure 1.25(a) and (b) are periodic, while the signal in Figure 1.25(c) is not.

Using the calculations that we have just made, we can also determine the fundamental period and frequency of discrete-time complex exponentials, where we define the fundamental frequency of a discrete-time periodic signal as we did in continuous time. That is, if $x[n]$ is periodic with fundamental period N, its fundamental frequency is $2\pi/N$. Consider, then, a periodic complex exponential $x[n] = e^{j\omega_0 n}$ with $\omega_0 \neq 0$. As we have just seen, ω_0 must satisfy eq. (1.56) for some pair of integers m and N, with $N > 0$. In Problem 1.35, it is shown that if $\omega_0 \neq 0$ and if N and m have no factors in common, then the fundamental period of $x[n]$ is N. Using this fact together with eq. (1.56), we find that the fundamental frequency of the periodic signal $e^{j\omega_0 n}$ is

$$\frac{2\pi}{N} = \frac{\omega_0}{m}. \tag{1.57}$$

Note that the fundamental period can also be written as

$$N = m\left(\frac{2\pi}{\omega_0}\right). \tag{1.58}$$

These last two expressions again differ from their continuous-time counterparts. In Table 1.1, we have summarized some of the differences between the continuous-time signal $e^{j\omega_0 t}$ and the discrete-time signal $e^{j\omega_0 n}$. Note that, as in the continuous-time case, the constant discrete-time signal resulting from setting $\omega_0 = 0$ has a fundamental frequency of zero, and its fundamental period is undefined.

TABLE 1.1   Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

*Assumes that m and N do not have any factors in common.

To gain some additional insight into these properties, let us examine again the signals depicted in Figure 1.25. First, consider the sequence $x[n] = \cos(2\pi n/12)$, depicted in Figure 1.25(a), which we can think of as the set of samples of the continuous-time sinusoid $x(t) = \cos(2\pi t/12)$ at integer time points. In this case, $x(t)$ is periodic with fundamental period 12 and $x[n]$ is also periodic with fundamental period 12. That is, the values of $x[n]$ repeat every 12 points, exactly in step with the fundamental period of $x(t)$.